Harvey Mudd College Math Tutorial:

The Gram-Schmidt Algorithm

In any **inner product space**, we can choose the basis in which to work. It often greatly simplifies calculations to work in an **orthogonal** basis. For one thing, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, then it is a simple matter to express any vector $\mathbf{w} \in V$ as a linear combination of the vectors in S:

$$w = \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{w}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n.$$

That is, $\mathbf{w}$ has coordinates

$$\begin{bmatrix} \frac{\langle \mathbf{w}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 \\ \frac{\langle \mathbf{w}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 \\ \vdots \\ \frac{\langle \mathbf{w}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \end{bmatrix}$$

relative to the basis S.

Given an arbitrary basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for an n-dimensional inner product space V, the **Gram-Schmidt algorithm** constructs an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for V:

Step 1 Let $\mathbf{v}_1 = \mathbf{u}_1$.

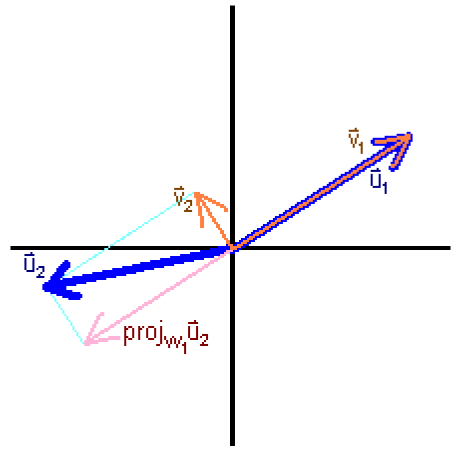
Step 2 Let $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$ where W_1 is the space spanned by $\mathbf{v}_1$, and $\text{proj}_{W_1}\mathbf{u}_2$ is the **orthogonal projection** of $\mathbf{u}_2$ on W_1.

Step 3 Let $\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$ where W_2 is the space spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Step 4 Let $\mathbf{v}_4 = \mathbf{u}_4 - \text{proj}_{W_3}\mathbf{u}_4 = \mathbf{u}_4 - \frac{\langle \mathbf{u}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \frac{\langle \mathbf{u}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2}\mathbf{v}_3$ where W_3 is the space spanned by $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$.

$\vdots$

Continue this process up to $\mathbf{v}_n$. The resulting orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ consists of n linearly independent vectors in V and so forms an orthogonal basis for V.



Notes

- To obtain an **orthonormal** basis for an inner product space V, use the Gram-Schmidt algorithm to construct an orthogonal basis. Then simply normalize each vector in the basis.
- For R^n with the Eudlidean inner product (dot product), we of course already know of the orthonormal basis $\{(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$. For more abstract spaces, however, the existence of an orthonormal basis is not obvious. The Gram-Schmidt algorithm is powerful in that it not only guarantees the existence of an orthonormal basis for any inner product space, but actually gives the construction of such a basis.

Example

Let $V = R^3$ with the Euclidean inner product. We will apply the Gram-Schmidt algorithm to orthogonalize the basis $\{(1,-1,1),(1,0,1),(1,1,2)\}$.

Step 1 $\mathbf{v}_1 = (1,-1,1)$.

Step 2

$$\begin{aligned}\mathbf{v}_2 &= (1,0,1) - \tfrac{(1,0,1)\cdot(1,-1,1)}{\|(1,-1,1)\|^2}(1,-1,1) \\ &= (1,0,1) - \tfrac{2}{3}(1,-1,1) \\ &= (\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{3}).\end{aligned}$$

Step 3

$$\begin{aligned}\mathbf{v}_3 &= (1,1,2) - \tfrac{(1,1,2)\cdot(1,-1,1)}{\|(1,-1,1)\|^2}(1,-1,1) - \tfrac{(1,1,2)\cdot(\frac{1}{3},\frac{2}{3},\frac{1}{3})}{\|(\frac{1}{3},\frac{2}{3},\frac{1}{3})\|^2}(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{3}) \\ &= (1,1,2) - \tfrac{2}{3}(1,-1,1) - \tfrac{5}{2}(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{3}) \\ &= (\tfrac{-1}{2},0,\tfrac{1}{2}).\end{aligned}$$

You can verify that $\left\{(1,-1,1),(\frac{1}{3},\frac{2}{3},\frac{1}{3}),(\frac{-1}{2},0,\frac{1}{2})\right\}$ forms an orthogonal basis for R^3. Normalizing the vectors in the orthogonal basis, we obtain the orthonormal basis

$$\left\{\left(\frac{\sqrt{3}}{3},\frac{-\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right),\left(\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right),\left(\frac{-\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\right\}.$$

Key Concepts

Given an arbitrary basis $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}$ for an n-dimensional inner product space V, the **Gram-Schmidt algorithm** constructs an orthogonal basis $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ for V:

Step 1 Let $\mathbf{v}_1=\mathbf{u}_1$.

Step 2 Let $\mathbf{v}_2=\mathbf{u}_2-\frac{\langle\mathbf{u}_2,\mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$.

Step 3 Let $\mathbf{v}_3=\mathbf{u}_3-\frac{\langle\mathbf{u}_3,\mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1-\frac{\langle\mathbf{u}_3,\mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$.

$\vdots$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]