Harvey Mudd College Math Tutorial:

Integration by Parts

We will use the **Product Rule** for derivatives to derive a powerful integration formula:

- Start with $(f(x)g(x))' = f(x)g'(x) + f'(x)g(x)$.
- Integrate both sides to get $f(x)g(x) = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$. (We need not include a constant of integration on the left, since the integrals on the right will also have integration constants.)
- Solve for $\int f(x)g'(x)\,dx$, obtaining

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

This formula frequently allows us to compute a difficult integral by computing a much simpler integral. We often express the Integration by Parts formula as follows:

Let

$$\begin{array}{ll} u = f(x) & dv = g'(x)\,dx \\ du = f'(x)\,dx & v = g(x) \end{array}$$

Then the formula becomes

$$\int u\,dv = uv - \int v\,du.$$

To integrate by parts, strategically choose u, dv and then apply the formula.

Example

Let's evaluate $\int xe^x\,dx$.

Let

$$\begin{array}{ll} u = x & dv = e^x\,dx \\ du = dx & v = e^x \end{array}$$

Then by integration by parts,

$$\begin{aligned} \int xe^x &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C. \end{aligned}$$

A Faulty Choice   A Reduction Formula

Integration by parts "works" on definite integrals as well:

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du.$$

Example

We will evaluate $\int_0^1 \arctan(x)\, dx$.

Let

$$u = \arctan(x) \qquad dv = dx$$
$$du = \frac{1}{1+x^2}\, dx \qquad v = x$$

Then by integration by parts,

$$\begin{aligned}
\int_0^1 \arctan(x) &= x\arctan(x)|_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx \\
&= x\arctan(x)|_0^1 - \frac{1}{2}\ln(1+x^2)\Big|_0^1 \\
&= \left(\frac{\pi}{4} - 0\right) - \left(\frac{1}{2}\ln(2) - 0\right) \\
&= \frac{\pi}{4} - \ln(\sqrt{2}).
\end{aligned}$$

Sometimes it is necessary to integrate twice by parts in order to compute an integral:

Example

Let's compute $\int e^x \cos x\, dx$.

Let

$$u = e^x \qquad dv = \cos x\, dx$$
$$du = e^x\, dx \qquad v = \sin x$$

Then $\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$.

It is not clear yet that we've accomplished anything, but now let's integrate the integral on the right-hand side by parts:

Now let

$$u = e^x \qquad dv = \sin x\, dx$$
$$du = e^x\, dx \qquad v = -\cos x$$

So $\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx$.

Substituting this into $\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$,

$$\begin{aligned}
\int e^x \cos x\, dx &= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\, dx\right] \\
&= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx.
\end{aligned}$$

The integal $\int e^x \cos x \, dx$ appears on both sides on the equation, so we can solve for it:

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x.$$

Finally,

$$\int e^x \cos x \, dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C.$$

Check by Differentiating

Key Concept

$$\int u \, dv = uv - \int v \, du.$$

- Choose u, dv in such a way that:
  1. u is easy to *differentiate*.
  2. dv is easy to *integrate*.
  3. $\int v \, du$ is easier to compute that $\int u \, dv$.
- Sometimes it is necessary to integrate by parts more than once.

[I'm ready to take the quiz.] [I need to review more.]
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