Harvey Mudd College Math Tutorial:
Integration by Parts

We will use the **Product Rule** for derivatives to derive a powerful integration formula:

- Start with \((f(x)g(x))' = f(x)g'(x) + f'(x)g(x)\).

- Integrate both sides to get \(f(x)g(x) = \int f(x)g'(x) \, dx + \int f'(x)g(x) \, dx\). (We need not include a constant of integration on the left, since the integrals on the right will also have integration constants.)

- Solve for \(\int f(x)g'(x) \, dx\), obtaining

\[
\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.
\]

This formula frequently allows us to compute a difficult integral by computing a much simpler integral. We often express the Integration by Parts formula as follows:

Let

\[
\begin{align*}
   u &= f(x) \\
   dv &= g'(x) \, dx \\
   du &= f'(x) \, dx \\
   v &= g(x)
\end{align*}
\]

Then the formula becomes

\[
\int u \, dv = uv - \int v \, du.
\]

To integrate by parts, strategically choose \(u, \ dv\) and then apply the formula.

**Example**

Let’s evaluate \(\int xe^x \, dx\).

Let

\[
\begin{align*}
   u &= x \\
   dv &= e^x \, dx \\
   du &= dx \\
   v &= e^x
\end{align*}
\]

Then by integration by parts,

\[
\int xe^x = xe^x - \int e^x \, dx = xe^x - e^x + C.
\]

**A Faulty Choice**

**A Reduction Formula**

Integration by parts “works” on definite integrals as well:

\[
\int_a^b u \, dv = uv|_a^b - \int_a^b v \, du.
\]
Example
We will evaluate $\int_0^1 \arctan(x) \, dx$.

Let
\[ u = \arctan(x) \quad dv = dx \]
\[ du = \frac{1}{1 + x^2} \, dx \quad v = x \]

Then by integration by parts,
\[ \int_0^1 \arctan(x) = x \arctan(x)|_0^1 - \int_0^1 \frac{x}{1 + x^2} \, dx \]
\[ = x \arctan(x)|_0^1 - \frac{1}{2} \ln(1 + x^2)|_0^1 \]
\[ = \left( \frac{\pi}{4} - 0 \right) - \left( \frac{1}{2} \ln(2) - 0 \right) \]
\[ = \frac{\pi}{4} - \ln(\sqrt{2}). \]

Sometimes it is necessary to integrate twice by parts in order to compute an integral:

Example
Let’s compute $\int e^x \cos x \, dx$.

Let
\[ u = e^x \quad dv = \cos x \, dx \]
\[ du = e^x \, dx \quad v = \sin x \]

Then $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$.

It is not clear yet that we’ve accomplished anything, but now let’s integrate the integral on the right-hand side by parts:

Now let
\[ u = e^x \quad dv = \sin x \, dx \]
\[ du = e^x \, dx \quad v = -\cos x \]

So $\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$.

Substituting this into $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$,
\[ \int e^x \cos x \, dx = e^x \sin x - \left[ -e^x \cos x + \int e^x \cos x \, dx \right] \]
\[ = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \]
The integral \( \int e^x \cos x \, dx \) appears on both sides on the equation, so we can solve for it:

\[
2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x.
\]

Finally,

\[
\int e^x \cos x \, dx = \frac{1}{2} e^x \sin x + \frac{1}{2} e^x \cos x + C.
\]

Check by Differentiating

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**Key Concept**

\[
\int u \, dv = uv - \int v \, du.
\]

- Choose \( u, \, dv \) in such a way that:
  1. \( u \) is easy to differentiate.
  2. \( dv \) is easy to integrate.
  3. \( \int v \, du \) is easier to compute than \( \int u \, dv \).

- Sometimes it is necessary to integrate by parts more than once.

[I’m ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]