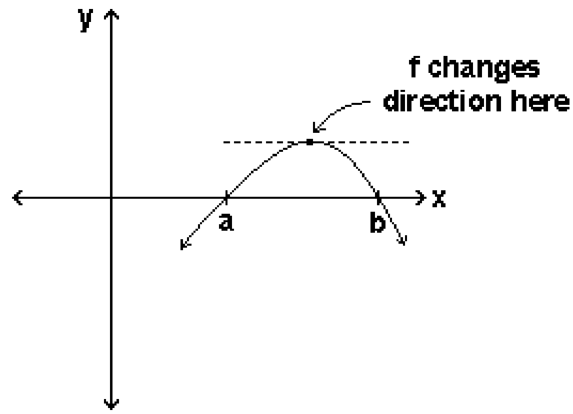
Harvey Mudd College Math Tutorial:

The Mean Value Theorem

We begin with a common-sense geometrical fact:

somewhere between two zeros of a non-constant continuous function f, the function must change direction



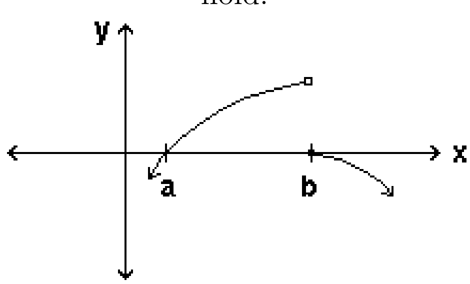
For a *differentiable* function, the derivative is 0 at the point where f changes direction. Thus, we expect there to be a point c where the tangent is horizontal. These ideas are precisely stated by Rolle's Theorem:

Rolle's Theorem

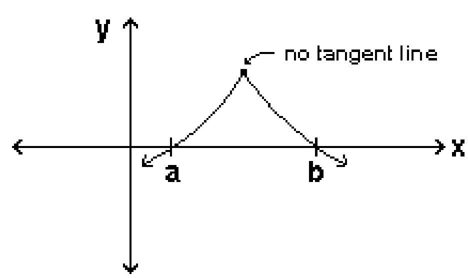
Let f be differentiable on (a, b) and continuous on $[a, b]$. If $f(a) = f(b) = 0$, then there is at least one point c in (a, b) for which $f'(c) = 0$.

Notice that both conditions on f are necessary. Without either one, the statement is false!

For a *discontinuous* function, the conclusion of Rolle's Theorem may not hold:



For a *continuous, non-differentiable* function, again this might not be the case:

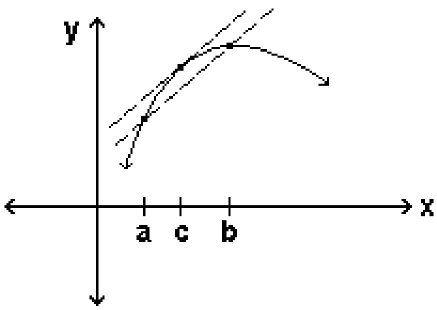


Though the theorem seems logical, we cannot be sure that it is always true without a proof.

Proof of Rolle's Theorem

The Mean Value Theorem is a generalization of Rolle's Theorem:

We now let $f(a)$ and $f(b)$ have values other than 0 and look at the secant line through $(a, f(a))$ and $(b, f(b))$. We expect that somewhere between a and b there is a point c where the tangent is parallel to this secant.



In Rolle's Theorem, the secant was horizontal so we looked for a horizontal tangent.

That is, the slopes of these two lines are equal. This is formalized in the Mean Value Theorem.

Mean Value Theorem

Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there is at least one point c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Here, $f'(c)$ is the slope of the tangent at c, while $\frac{f(b) - f(a)}{b - a}$ is the slope of the secant through a and b. Intuitively, we see that if we translate the secant line in the figure upwards, it will eventually just touch the curve at the single point c and will be tangent at c. However, basing conclusions on a single example can be disastrous, so we need a proof.

Proof of the Mean Value Theorem

Consequences of the Mean Value Theorem

The Mean Value Theorem is behind many of the important results in calculus. The following statements, in which we assume f is differentiable on an open interval I, are consequences of the Mean Value Theorem:

- $f'(x) = 0$ everywhere on I if and only if f is constant on I.
- If $f'(x) = g'(x)$ for all x on I, then f and g differ at most by a constant on I.
- If $f'(x) > 0$ for all x on I, then f is *increasing* on I.

  If $f'(x) < 0$ for all x on I, then f is *decreasing* on I.

Key Concepts

Mean Value Theorem

Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there is at least one point c in (a, b) for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]