Proof of the Mean Value Theorem

The equation of the secant through \((a, f(a))\) and \((b, f(b))\) is

\[
y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)
\]

which we can rewrite as

\[
y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).
\]

Let

\[
g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].
\]

Note that \(g(a) = g(b) = 0\). Also, \(g\) is continuous on \([a, b]\) and differentiable on \((a, b)\) since \(f\) is. So by Rolle’s Theorem there exists \(c\) in \((a, b)\) such that \(g'(c) = 0\).

But \(g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}\), so

\[
g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0.
\]

Therefore,

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

and the proof is complete.