Harvey Mudd College Math Tutorial:

Partial Fractions

Consider the integral

$$\int \frac{3x^3 - 2x^2 - 19x - 7}{x^2 - x - 6}\, dx.$$

The integrand is an **improper rational function**. By "long division" of polynomials, we can rewrite the integrand as the sum of a polynomial and a **proper rational function** "remainder":

$$\frac{3x^3 - 2x^2 - 19x - 7}{x^2 - x - 6} = 3x + 1 + \frac{-1}{x^2 - x - 6}.$$

So

$$\int \frac{3x^3 - 2x^2 - 19x - 7}{x^2 - x - 6}\, dx = \int \left(3x + 1 + \frac{-1}{x^2 - x - 6}\right) dx.$$

This looks much easier to work with! We can integrate $3x + 1$ immediately, but what about $\frac{-1}{x^2 - x - 6}$?

Notice that

$$\frac{-1}{x^2 - x - 6} = \frac{-1}{(x+2)(x-3)}$$

which suggests that we try to write $\frac{-1}{x^2 - x - 6}$ as the sum of two rational functions of the form $\frac{A}{x+2}$ and $\frac{B}{x-3}$:

$$\frac{-1}{x^2 - x - 6} = \frac{A}{x+2} + \frac{B}{x-3}.$$

This is called the **Partial Fraction Decomposition** for $\frac{-1}{x^2 - x - 6}$.

Our goal now is to determine A and B. Multiplying both sides of the equation by $(x+2)(x-3)$ to clear the fractions,

$$-1 = A(x-3) + B(x+2).$$

There are two methods for solving for A and B:

Method 1

Collect like terms on the right:
$-1 = (A+B)x + (-3A + 2B)$.
Now equate coefficients of corresponding powers of x:
$A + B = 0, \quad -3A + 2B = -1$.
Solving this system,
$A = 1/5$, $B = -1/5$.

Method 2

The equation holds for *all* x.
Let $x = -2$:
$-1 = A(-2-3) + B(-2+2)$
$-1 = -5A \qquad \longrightarrow A = 1/5$.
Now let $x = 3$:
$-1 = A(3-3) + B(3+2)$
$-1 = 5B \qquad \longrightarrow B = -1/5$.

So

$$\frac{-1}{x^2-x-6}=\frac{\frac{1}{5}}{x+2}-\frac{\frac{1}{5}}{x-3}.$$

Returning to the original integral,

$$\begin{aligned}\int \frac{3x^3-2x^2-19x-7}{x^2-x-6}\,dx &= \int\left(3x+1+\frac{\frac{1}{5}}{x+2}-\frac{\frac{1}{5}}{x-3}\right)dx\\ &= \frac{3}{2}x^2+x+\frac{1}{5}\ln\left|\frac{x+2}{x-3}\right|+C.\end{aligned}$$

In the next example, we have repeated factors in the denominator, as well as an **irreducible quadratic factor**.

Example

We will evaluate

$$\int \frac{x-1}{x^2(x^2+x+1)}\,dx.$$

The integrand is a proper rational function, which we would like to decompose into proper rational functions of the form

$$\frac{A}{x},\quad \frac{B}{x^2},\quad \text{and}\quad \frac{Cx+D}{x^2+x+1}.$$

[Notice that we have two factors of x in the denominator of the integrand, leading to terms of the form $\frac{A}{x}$ and $\frac{B}{x^2}$ in the decomposition. The factor x^2+x+1 is irreducible and quadratic, so any proper rational function with x^2+x+1 as denominator has the form $\frac{Cx+D}{x^2+x+1}$ where C or D may be 0.]

Set

$$\frac{x-1}{x^2(x^2+x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+x+1}.$$

Multiplying through by $x^2(x^2+x+1)$,

$$x-1=Ax(x^2+x+1)+B(x^2+x+1)+(Cx+D)x^2.$$

Since x^2+x+1 has no real roots, it is easiest to solve for A and B using Method 1:

Collecting like terms on the right,

$$x-1=(A+C)x^3+(A+B+D)x^2+(A+B)x+B.$$

Equating corresponding powers of x,

$$\left.\begin{aligned}A+C &= 0\\ A+B+D &= 0\\ A+B &= 1\\ B &= -1\end{aligned}\right\}\longrightarrow\quad \begin{aligned}A&=2\\ B&=-1\\ C&=-2\\ D&=-1\end{aligned}\quad\longrightarrow\quad \frac{2}{x}-\frac{1}{x^2}-\frac{2x+1}{x^2+x+1}.$$

So

$$
\begin{aligned}
\frac{x-1}{x^2(x^2+x+1)}\,dx &= \int \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2x+1}{x^2+x+1}\right) dx \\
&= 2\ln|x| + \frac{1}{x} - \ln|x^2+x+1| + C \\
&= \frac{1}{x} + \ln\left|\frac{x^2}{x^2+x+1}\right| + C.
\end{aligned}
$$

Key Concepts

Partial Fraction Decomposition of a Rational Function

- If the rational function is improper, use “long division” of polynomials to write it as the sum of a polynomial and a proper rational function “remainder.”
- Decompose the proper rational function as a sum of rational functions of the form

  $$\frac{A}{(x-\alpha)^k} \quad \text{and} \quad \frac{Bx+C}{(x^2+\beta x+\gamma)^k} \quad (x^2+\beta x+\gamma \text{ irreducible})$$

  where:

  - Each factor $(x-\alpha)^m$ in the denominator of the proper rational function suggests terms
    $$\frac{A_1}{(x-\alpha)} + \frac{A_2}{(x-\alpha)^2} + \ldots + \frac{A_m}{(x-\alpha)^m}.$$
  - Each factor $(x^2+\beta x+\gamma)^n$ suggests terms
    $$\frac{B_1x+C_1}{(x^2+\beta x+\gamma)} + \frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2} + \ldots + \frac{B_nx+C_n}{(x^2+\beta x+\gamma)^n}.$$
- Determine the (unique) values of all the constants involved.
  - Use either Method 1 or Method 2, or a combination of both.

The partial fraction decomposition is often used to rewrite a complicated rational function integrand as a sum of terms, each of which is straightforward to integrate.

[I’m ready to take the quiz.] [I need to review more.]
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