Consider the integral
\[ \int \frac{3x^3 - 2x^2 - 19x - 7}{x^2 - x - 6} \, dx. \]
The integrand is an improper rational function. By “long division” of polynomials, we can rewrite the integrand as the sum of a polynomial and a proper rational function “remainder”:
\[ \frac{3x^3 - 2x^2 - 19x - 7}{x^2 - x - 6} = 3x + 1 + \frac{-1}{x^2 - x - 6}. \]
So
\[ \int \frac{3x^3 - 2x^2 - 19x - 7}{x^2 - x - 6} \, dx = \int \left( 3x + 1 + \frac{-1}{x^2 - x - 6} \right) \, dx. \]
This looks much easier to work with! We can integrate \( 3x + 1 \) immediately, but what about \( \frac{-1}{x^2 - x - 6} \)?

Notice that
\[ \frac{-1}{x^2 - x - 6} = \frac{-1}{(x + 2)(x - 3)} \]
which suggests that we try to write \( \frac{-1}{x^2 - x - 6} \) as the sum of two rational functions of the form \( \frac{A}{x + 2} \) and \( \frac{B}{x - 3} \):
\[ \frac{-1}{x^2 - x - 6} = \frac{A}{x + 2} + \frac{B}{x - 3}. \]
This is called the Partial Fraction Decomposition for \( \frac{-1}{x^2 - x - 6} \).

Our goal now is to determine \( A \) and \( B \). Multiplying both sides of the equation by \((x+2)(x-3)\) to clear the fractions,
\[ -1 = A(x - 3) + B(x + 2). \]
There are two methods for solving for \( A \) and \( B \):

**Method 1**
Collect like terms on the right:
\[ -1 = (A + B)x + (-3A + 2B). \]
Now equate coefficients of corresponding powers of \( x \):
\[ A + B = 0, \quad -3A + 2B = -1. \]
Solving this system,
\[ A = 1/5, \quad B = -1/5. \]

**Method 2**
The equation holds for all \( x \). Let \( x = -2 \):
\[ -1 = A(-2 - 3) + B(-2 + 2) \]
\[ -1 = -5A \quad \longrightarrow \quad A = 1/5. \]
Now let \( x = 3 \):
\[ -1 = A(3 - 3) + B(3 + 2) \]
\[ -1 = 5B \quad \longrightarrow \quad B = -1/5. \]
So
\[ \frac{-1}{x^2 - x - 6} = \frac{1}{5} \frac{1}{x + 2} - \frac{1}{5} \frac{1}{x - 3}. \]

Returning to the original integral,
\[ \int \frac{3x^3 - 2x^2 - 19x - 7}{x^2 - x - 6} \, dx = \int \left( 3x + 1 + \frac{1}{5} \frac{1}{x + 2} - \frac{1}{5} \frac{1}{x - 3} \right) \, dx \]
\[ = \frac{3}{2} x^2 + x + \frac{1}{5} \ln \left| \frac{x + 2}{x - 3} \right| + C. \]

In the next example, we have repeated factors in the denominator, as well as an **irreducible quadratic factor**.

**Example**

We will evaluate
\[ \int \frac{x - 1}{x^2(x^2 + x + 1)} \, dx. \]

The integrand is a proper rational function, which we would like to decompose into proper rational functions of the form
\[ \frac{A}{x}, \quad \frac{B}{x^2}, \quad \text{and} \quad \frac{C x + D}{x^2 + x + 1}. \]

[Notice that we have two factors of \( x \) in the denominator of the integrand, leading to terms of the form \( \frac{A}{x} \) and \( \frac{B}{x^2} \) in the decomposition. The factor \( x^2 + x + 1 \) is irreducible and quadratic, so any proper rational function with \( x^2 + x + 1 \) as denominator has the form \( \frac{C x + D}{x^2 + x + 1} \), where \( C \) or \( D \) may be 0.]

Set
\[ \frac{x - 1}{x^2(x^2 + x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C x + D}{x^2 + x + 1}. \]

Multiplying through by \( x^2(x^2 + x + 1) \),
\[ x - 1 = Ax(x^2 + x + 1) + B(x^2 + x + 1) + (Cx + D)x^2. \]

Since \( x^2 + x + 1 \) has no real roots, it is easiest to solve for \( A \) and \( B \) using Method 1:

Collecting like terms on the right,
\[ x - 1 = (A + C)x^3 + (A + B + D)x^2 + (A + B)x + B. \]

Equating corresponding powers of \( x \),
\[ \begin{align*}
A + C &= 0 \\
A + B + D &= 0 \\
A + B &= 1 \\
B &= -1
\end{align*} \]
\[ \rightarrow \quad \begin{align*}
A &= 2 \\
B &= -1 \\
C &= -2 \\
D &= -1
\end{align*} \]
\[ \rightarrow \quad \frac{2}{x} - \frac{1}{x^2} - \frac{2x + 1}{x^2 + x + 1}. \]
So

\[
\frac{x - 1}{x^2(x^2 + x + 1)} \, dx = \int \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2x + 1}{x^2 + x + 1}\right) \, dx
\]

\[
= 2 \ln|x| + \frac{1}{x} - \ln|x^2 + x + 1| + C
\]

\[
= \frac{1}{x} + \ln\left|\frac{x^2}{x^2 + x + 1}\right| + C.
\]

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**Key Concepts**

**Partial Fraction Decomposition of a Rational Function**

- If the rational function is improper, use “long division” of polynomials to write it as the sum of a polynomial and a proper rational function “remainder.”

- Decompose the proper rational function as a sum of rational functions of the form

  \[
  \frac{A}{(x - \alpha)^k} \quad \text{and} \quad \frac{Bx + C}{(x^2 + \beta x + \gamma)^k} \quad (x^2 + \beta x + \gamma \text{ irreducible})
  \]

  where:

  - Each factor \((x - \alpha)^m\) in the denominator of the proper rational function suggests terms

    \[
    \frac{A_1}{x - \alpha} + \frac{A_2}{(x - \alpha)^2} + \ldots + \frac{A_m}{(x - \alpha)^m}.
    \]

  - Each factor \((x^2 + \beta x + \gamma)^n\) suggests terms

    \[
    \frac{B_1 x + C_1}{x^2 + \beta x + \gamma} + \frac{B_2 x + C_2}{(x^2 + \beta x + \gamma)^2} + \ldots + \frac{B_n x + C_n}{(x^2 + \beta x + \gamma)^n}.
    \]

- Determine the (unique) values of all the constants involved.

  - Use either Method 1 or Method 2, or a combination of both.

The partial fraction decomposition is often used to rewrite a complicated rational function integrand as a sum of terms, each of which is straightforward to integrate.