Harvey Mudd College Math Tutorial:

Quotient Rule for Derivatives

Suppose we are working with a function $h(x)$ that is a ratio of two functions $f(x)$ and $g(x)$.

How is the derivative of $h(x)$ related to $f(x)$, $g(x)$, and their derivatives?

Quotient Rule

Let f and g be differentiable at x with $g(x) \neq 0$. Then f/g is differentiable at x and

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof

Examples

- If $f(x) = \dfrac{2x+1}{x-3}$, then

$$\begin{aligned} f'(x) &= \frac{(x-3)\frac{d}{dx}[2x+1] - (2x+1)\frac{d}{dx}[x-3]}{[x-3]^2} \\ &= \frac{(x-3)(2) - (2x+1)(1)}{(x-3)^2} \\ &= -\frac{7}{(x-3)^2}. \end{aligned}$$

- If $f(x) = \tan x = \dfrac{\sin x}{\cos x}$, then

$$\begin{aligned} f'(x) &= \frac{\cos(x)\frac{d}{dx}[\sin(x)] - \sin(x)\frac{d}{dx}[\cos x]}{[\cos x]^2} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x), \end{aligned}$$

  verifying the familiar differentiation formula for $\tan(x)$.

- If $f(x) = \dfrac{1}{g(x)}$, then

$$f'(x) = \left[\frac{1}{g(x)}\right]' = \frac{g(x)\frac{d}{dx}[1] - (1)g'(x)}{[g(x)]^2} = \frac{g(x)(0) - (1)g'(x)}{[g(x)]^2} = -\frac{g'(x)}{[g(x)]^2}.$$

For example, $\dfrac{d}{dx}[x^{-4}] = \dfrac{d}{dx}\left[\dfrac{1}{x^4}\right] = -\dfrac{\frac{d}{dx}[x^4]}{[x^4]^2} = -\dfrac{4x^3}{x^8} = -\dfrac{4}{x^5} = -4x^{-5}$.

Key Concepts

Quotient Rule

Let f and g be differentiable at x with $g(x) \neq 0$. Then f/g is differentiable at x and

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

[I'm ready to take the quiz.] [I need to review more.]
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