Harvey Mudd College Math Tutorial:  
**Quotient Rule for Derivatives**

Suppose we are working with a function $h(x)$ that is a ratio of two functions $f(x)$ and $g(x)$.

**How is the derivative of $h(x)$ related to $f(x)$, $g(x)$, and their derivatives?**

**Quotient Rule**

Let $f$ and $g$ be differentiable at $x$ with $g(x) \neq 0$. Then $f/g$ is differentiable at $x$ and

$$
\left[ \frac{f(x)}{g(x)} \right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.
$$

**Proof**

**Examples**

- If $f(x) = \frac{2x + 1}{x - 3}$, then

  $$
f'(x) = \frac{(x - 3) \frac{d}{dx}[2x + 1] - (2x + 1) \frac{d}{dx}[x - 3]}{[x - 3]^2}
  = \frac{(x - 3)(2) - (2x + 1)(1)}{(x - 3)^2}
  = -\frac{7}{(x - 3)^2}.
$$

- If $f(x) = \tan x = \frac{\sin x}{\cos x}$, then

  $$
f'(x) = \frac{\cos(x) \frac{d}{dx}[\sin(x)] - \sin(x) \frac{d}{dx}[\cos x]}{[\cos x]^2}
  = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}
  = \frac{1}{\cos^2(x)}
  = \sec^2(x),
$$

  verifying the familiar differentiation formula for $\tan(x)$.  

• If \( f(x) = \frac{1}{g(x)} \), then

\[
f'(x) = \left[ \frac{1}{g(x)} \right]' = \frac{g(x) \frac{d}{dx} [1] - (1)g'(x)}{[g(x)]^2}
\]

\[
= \frac{g(x)(0) - (1)g'(x)}{[g(x)]^2}
\]

\[
= -\frac{g'(x)}{[g(x)]^2}.
\]

For example, \( \frac{d}{dx}[x^{-4}] = \frac{d}{dx}[1 - 4x^3] = -\frac{4}{x^4} = -\frac{4}{x^8} = -\frac{4}{x^5} = -4x^{-5} \).

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**Key Concepts**

**Quotient Rule**

Let \( f \) and \( g \) be differentiable at \( x \) with \( g(x) \neq 0 \). Then \( f/g \) is differentiable at \( x \) and

\[
\left[ \frac{f(x)}{g(x)} \right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.
\]