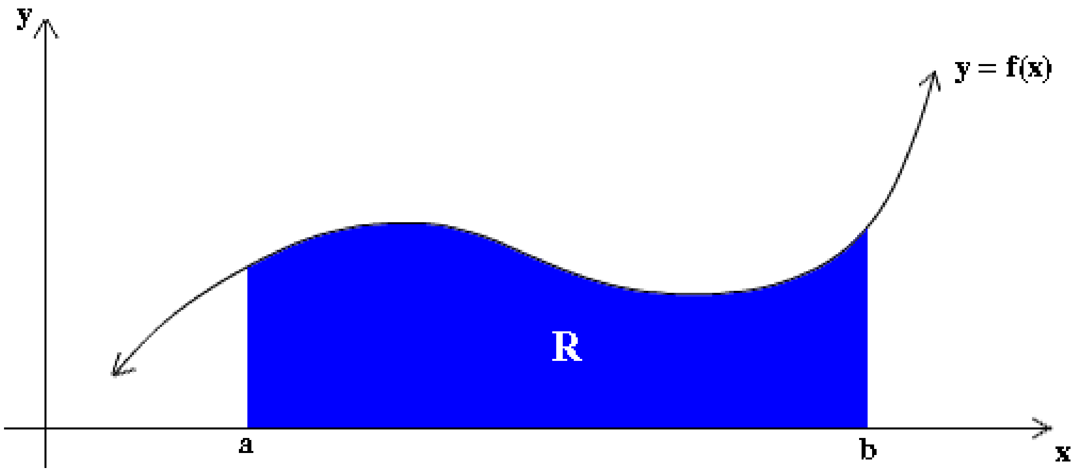
Harvey Mudd College Math Tutorial:

Riemann Sums

Suppose that a function f is continuous and non-negative on an interval $[a, b]$.

Let's compute the area of the region R bounded above by the curve $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$.

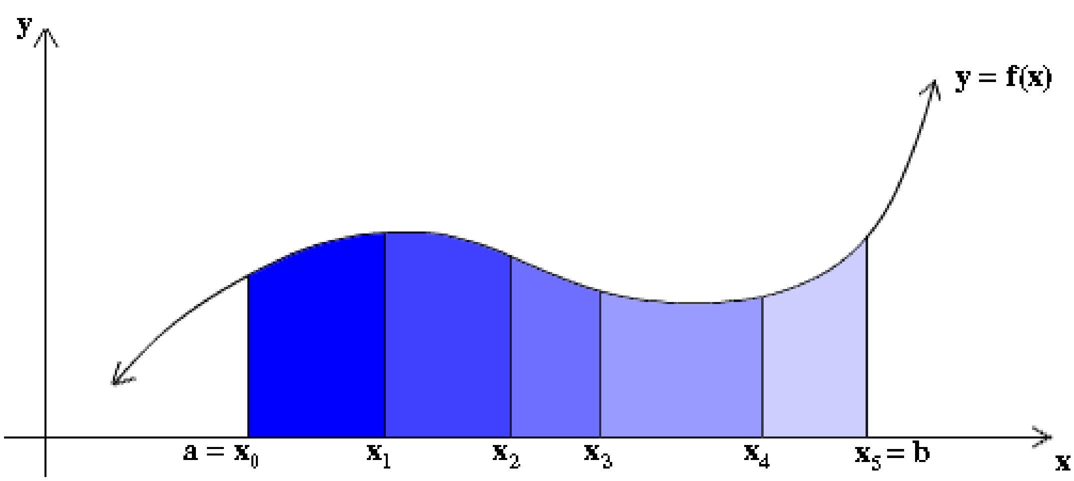


We will obtain this area as the limit of a sum of areas of rectangles as follows:

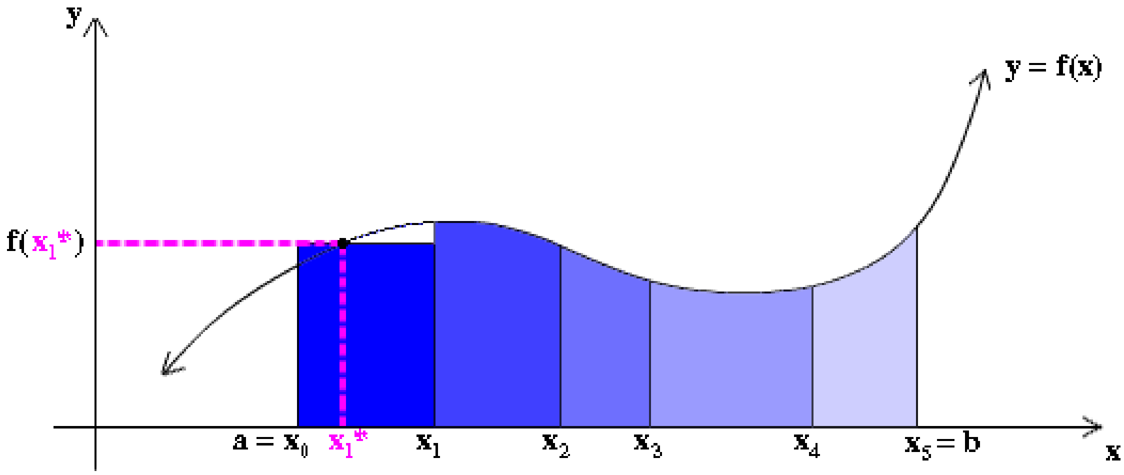
First, we will divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

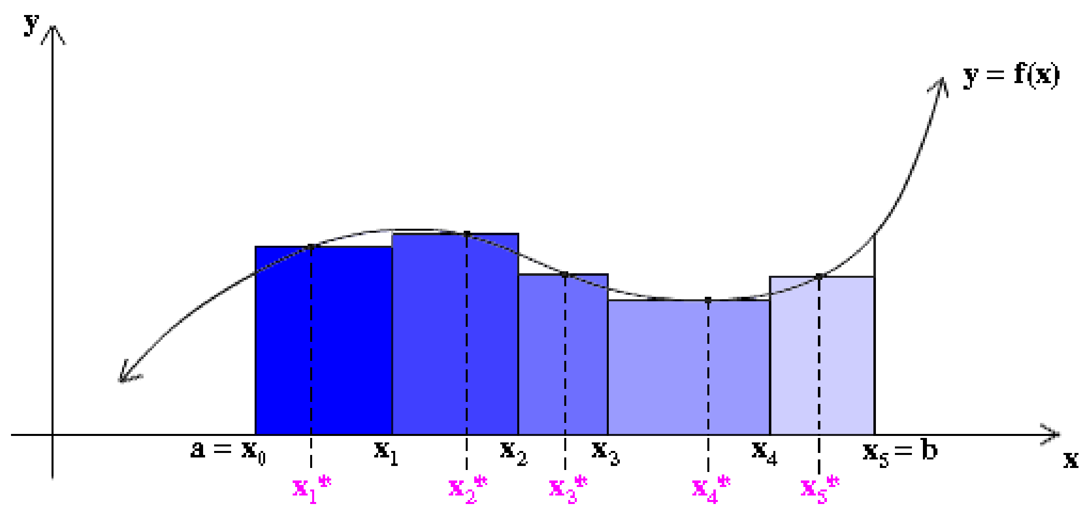
where $a = x_0 < x_1 < \ldots < x_n = b$. (This is called a **partition** of the interval.) The intervals need not all be the same length, so call the lengths of the intervals Δx_1, Δx_2, ..., Δx_n, respectively. This partition divides the region R into n strips.



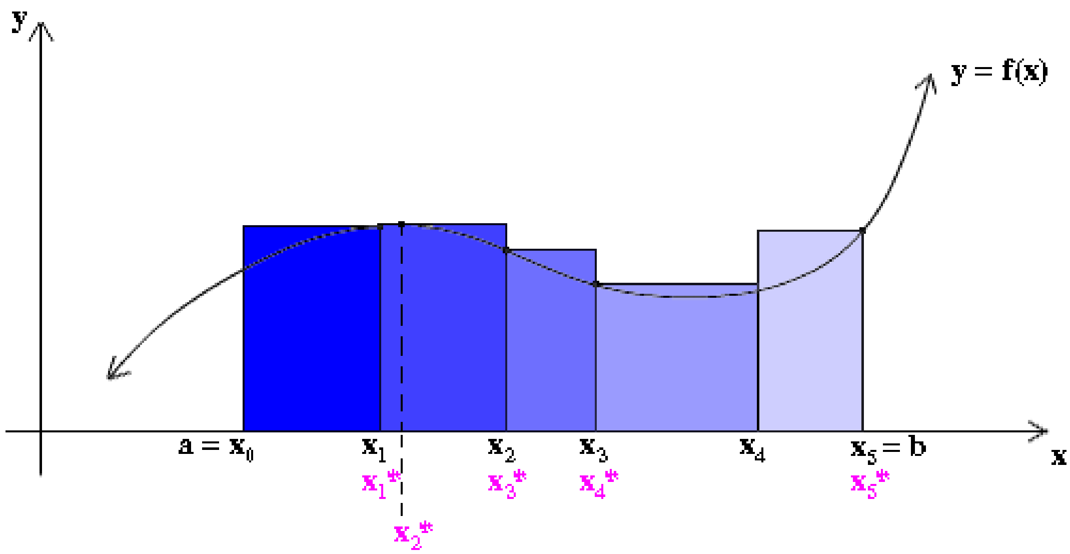
Next, let's approximate each strip by a rectangle with height equal to the height of the curve $y = f(x)$ at some arbitrary point in the subinterval. That is, for the first subinterval $[x_0, x_1]$, select some x_1^* contained in that subinterval and use $f(x_1^*)$ as the height of the first rectangle. The area of that rectangle is then $f(x_1^*)\Delta x_1$.



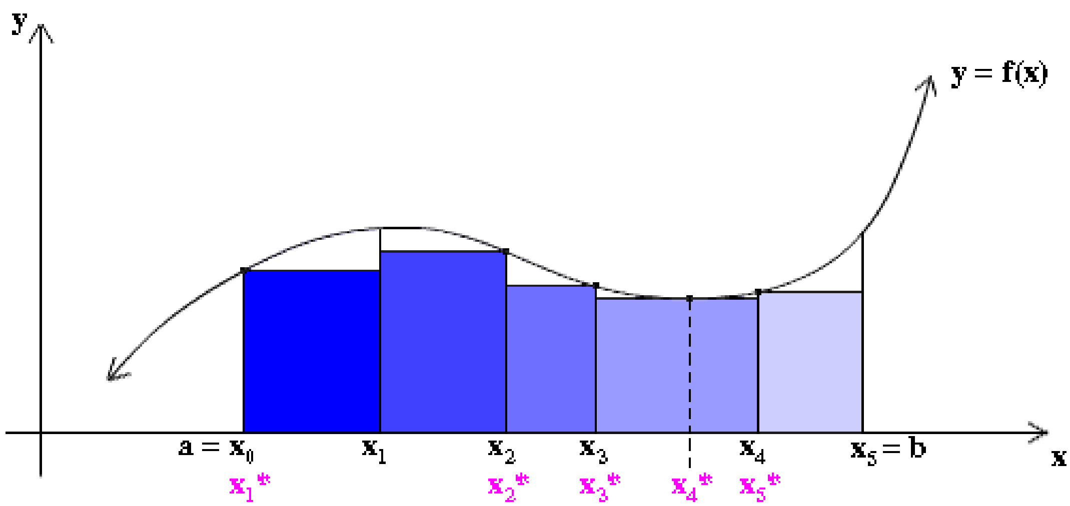
Similarly, for each subinterval $[x_{i-1}, x_i]$, we will choose some x_i^* and calculate the area of the corresponding rectangle to be $f(x_i^*)\Delta x_i$. The approximate area of the region R is then the sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ of these rectangles.



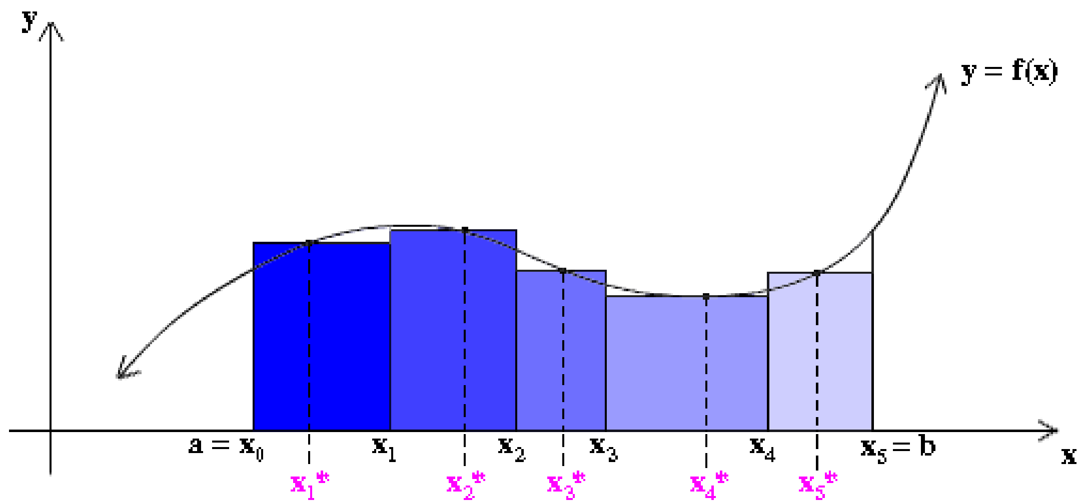
Depending on what points we select for the x_i^*, our estimate may be too large or too small. For example, if we choose each x_i^* to be the point in its subinterval giving the *maximum* height, we will *overestimate* the area of R. (This is called a **upper sum**.)



If, on the other hand, we choose each x_i^* to be the point in its subinterval giving the *mimimum* height, we will *underestimate* the area of R. (This is called a **lower sum**.)



When the points x_i^* are chosen randomly, the sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ is called a **Riemann Sum**



and will give an approximation for the area of R that is in between the lower and upper sums. The upper and lower sums may be considered specific Riemann sums.

As we decrease the widths of the rectangles, we expect to be able to approximate the area of R better. In fact, as max $\Delta x_i \to 0$, we get the exact area of R, which we denote by the definite integral $\int_a^b f(x)\,dx$. That is,

$$\int_a^b f(x)\,dx = \lim_{max \Delta x_i \to 0} \left(\sum_{i=1}^{n} f(x_i^*) \Delta x_i \right).$$

Notes

- This definition of the definite integral still holds if $f(x)$ assumes both positive and negative values on $[a, b]$. It even holds if $f(x)$ has finitely many discontinuities but is bounded.
- For a more rigorous treatment of Riemann sums, consult your calculus text.

The following Exploration allows you to approximate the area under various curves under the interval $[0, 5]$. You can create a partition of the interval and view an upper sum, a lower sum, or another Riemann sum using that partition. The Exploration will give you the exact area and calculate the area of your approximation. To create a partition, choose which type of sum you would like to see and click the mouse between the partition labels x_0 and x_1.

Exploration

Key Concept

Let f be defined on $[a, b]$ and let $x_0, x_1, \ldots, x_n$ be a partition of $[a, b]$.

For each $[x_{i-1}, x_i]$, let $x_i^* \in [x_{i-1}, x_i]$.

Then the definite integral of f over $[a, b]$ as defined as

$$\int_a^b f(x)\,dx = \lim_{max \Delta x_i \to 0} \left(\sum_{i=1}^{n} f(x_i^*) \Delta x_i \right).$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]