Harvey Mudd College Math Tutorial:

Concavity and the Second Derivative Test

You are learning that calculus is a valuable tool. One of the most important applications of differential calculus is to find **extreme function values**. The calculus methods for finding the maximum and minimum values of a function are the basic tools of **optimization theory**, a very active branch of mathematical research applied to nearly all fields of practical endeavor. Although modern optimization theory is considerably more advanced, its methods and fundamental ideas clearly show their historical relationship to the calculus. In this tutorial you will review how the second derivative of a function is related to the shape of its graph and how that information can be used to classify relative extreme values.

Some First Derivative Facts

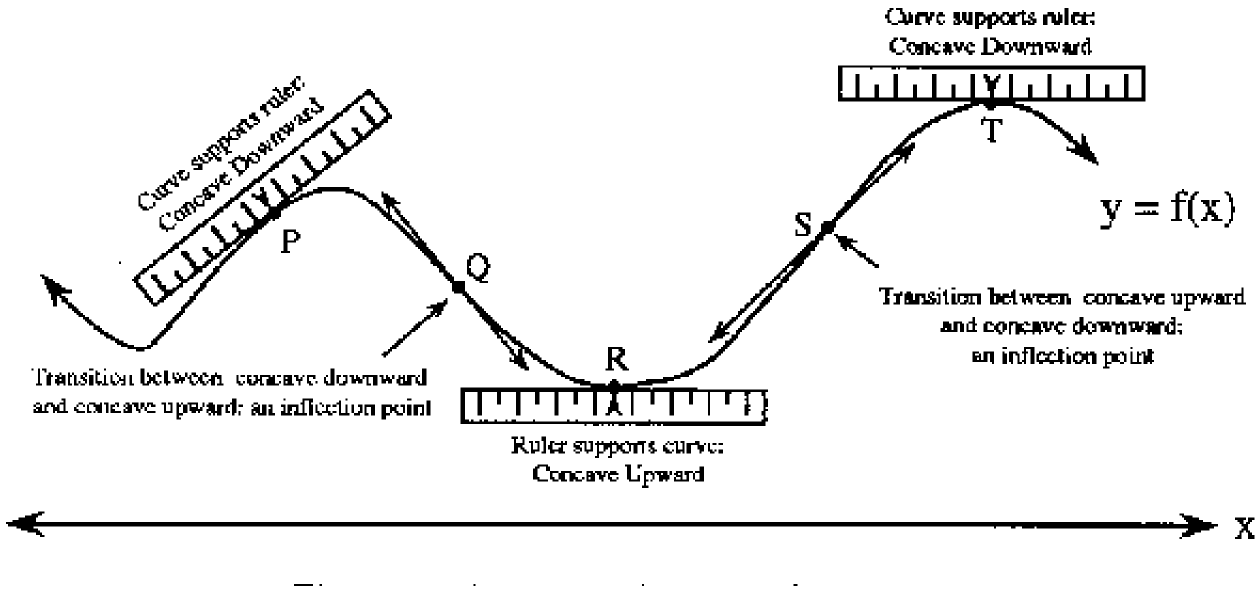
If you have not already done so, you should review the tutorial on the **First Derivative**. Click **here** to see a picture that summarizes the First Derivative Test.

Concavity

The Second Derivative Test provides a means of classifying relative extreme values by using the sign of the second derivative at the critical number. To appreciate this test, it is first necessary to understand the concept of concavity.

The graph of a function f is **concave upward** at the point $(c, f(c))$ if $f'(c)$ exists and if for all x in some open interval containing c, the point $(x, f(x))$ on the graph of f lies above the corresponding point on the graph of the tangent line to f at c. This is expressed by the inequality $f(x) > [f(c) + f'(c)(x - c)]$ for all x in some open interval containing c. Imagine holding a ruler along the tangent line through the point $(c, f(c))$: if the ruler *supports the graph of* f near $(c, f(c))$, then the graph of the function is concave upward.

The graph of a function f is **concave downward** at the point $(c, f(c))$ if $f'(c)$ exists and if for all x in some open interval containing c, the point $(x, f(x))$ on the graph of f lies below the corresponding point on the graph of the tangent line to f at c. This is expressed by the inequality $f(x) < [f(c) + f'(c)(x - c)]$ for all x in some open interval containing c. In this situation *the graph of* f *supports the ruler*. This is pictured below:



Concavity and the Second Derivative

The important result that relates the concavity of the graph of a function to its derivatives is the following one:

> CONCAVITY THEOREM: If the function f is twice differentiable at $x = c$, then the graph of f is concave upward at $(c, f(c))$ if $f''(c) > 0$ and concave downward if $f''(c) < 0$.

Example
Suppose $f(x) = x^3 - 3x^2 + x - 2$. Let's determine where the graph of f is concave up and where it is concave down. Since f is twice-differentiable for all x, we use the result given above and first determine that $f''(x) = 6(x - 1)$. Thus, $f''(x) > 0$ if $x > 1$ and $f''(x) < 0$ if $x < 1$. By the Concavity Theorem, the graph of f is concave up for $x > 1$ and concave down for $x < 1$.

Inflection Points

Notice in the example above, that the concavity of the graph of f *changes sign at* $x = 1$. Points on the graph of f where the concavity changes from up-to-down or down-to-up are called **inflection points** of the graph. The following result connects the concept of inflection points to the derivatives properties of the function:

INFLECTION POINT THEOREM: If $f'(c)$ exists and $f''(c)$ *changes sign at* $x = c$, then the point $(c, f(c))$ is an **inflection point** of the graph of f. If $f''(c)$ exists at the inflection point, then $f''(c) = 0$.

If we return to our example, where $f(x) = x^3 - 3x^2 + x - 2$, the INFLECTION POINT THEOREM verifies that the graph of f has an inflection point at $x = 1$, since $f''(1) = 0$.

The Second Derivative Test

The Second Derivative Test relates the concepts of critical points, extreme values, and concavity to give a very useful tool for determining whether a critical point on the graph of a function is a relative minimum or maximum.

THE SECOND DERIVATIVE TEST: Suppose that c is a critical point at which $f'(c) = 0$, that $f'(x)$ exists in a neighborhood of c, and that $f''(c)$ exists. Then f has a relative maximum value at c if $f''(c) < 0$ and a relative minimum value at c if $f''(c) > 0$. If $f''(c) = 0$, the test is not informative.

Example

Let's find and classify the extreme values for the function f with values $f(x) = x^3 - 3x^2 + x - 2$ that was introduced above. We find that $f'(x) = 3x^2 - 6x + 1$, and so there are two critical numbers where $f'(c) = 0$:

$$c_1 = 3 - \sqrt{6} \quad \text{and} \quad c_2 = 3 + \sqrt{6}.$$

Notice that $c_1 < 1$ and that $f''(c) < 0$. Thus f has a relative maximum at $x = 3 - \sqrt{6}$. Since $c_2 > 1$ and $f''(c_2) > 0$, the Second Derivative Test informs us that f has a relative minumum at $x = 3 + \sqrt{6}$.

Key Concepts

- **Concavity Theorem**:

  If the function f is twice differentiable at $x = c$, then the graph of f is concave upward at $(c, f(c))$ if $f''(c) > 0$ and concave downward if $f''(c) < 0$.

- **The Second Derivative Test**:

  Suppose that c is a critical point at which $f'(c) = 0$, that $f'(x)$ exists in a neighborhood of c, and that $f''(c)$ exists. Then f has a relative maximum value at c if $f''(c) < 0$ and a relative minimum value at c if $f''(c) > 0$. If $f''(c) = 0$, the test is not informative.

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]