Harvey Mudd College Math Tutorial

The Tangent Line Approximation

Suppose we want to find the tangent to a curve. Just how can we go about finding one? *Click here for the first applet.*

The following text is based on the applet found by clicking the above link or by going to the URL http://www.math.hmc.edu/calculus/tutorials/tangent_line/applet1.html. If you cannot open the applet then use a graphing program/calculator to plot the curve $f(x) = x^2 - 1$ *and follow along as best you can.*

Here is one way:

- Pick a point Q by clicking on the curve on the applet. (The line that appears is the secant line between P and Q.)
- Now drag point Q towards point P.

As Q approaches P, the secant line approximates the tangent line better and better. The limiting position of the secant line as Q approaches P is the tangent to the curve at P.

If the curve is given by $y = f(x)$ and P has the coordinates (x_0, y_0), then the slope of the tangent line at P is $f'(x_0)$, the derivative of f evaluated at x_0.

Let's find the equation of the line tangent to the parabola at $(2, 3)$.

- Drag point P to $(2, 3)$.
- Now pick another point Q on the parabola and drag Q towards P to find the tangent to the curve at P.

The slope of the tangent is just $f'(x)$ evaluated at x.

$$\begin{aligned} f'(x) &= 2x \\ f'(2) &= 4. \end{aligned}$$

Now, the equation of the line can be written in **point-slope form** like this:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - y_0 &= f'(x_0)(x - x_0) \\ y - 3 &= 4(x - 2) \end{aligned}$$

since the line passes through the point $(2, 3)$ and has slope 4.

In **slope-intercept form**, the equation of the tangent line becomes

$$y = 4x - 5.$$

- Drag P along the parabola or enter the x-coordinate for point P.

  Click here for the next applet or go to URL http://www.math.hmc.edu/calculus/tutorials/tangent_line/applet2.html. The function being studied in this applet is the same as in the previous applet.

- Notice how the equation of the tangent line changes as you move point P.

What happens when $x = 0$ for this function? What about as $|x|$ gets large?

Now that we can find the tangent to a curve at a point, of what use is this?

- "Magnify" the parabola by zooming in on point P.

Do you notice that as you zoom in on P the curve looks more and more linear and is approximated better and better by the tangent line?

Let's get more specific:

Near x_0, we saw that $y = f(x)$ can be approximated by the tangent line $y - y_0 = f'(x_0)(x - x_0)$. Writing this as $y = y_0 + f'(x_0)(x - x_0)$ and noting that $y = f(x_0)$, we find that

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0).}$$

(Notice that the right-hand side is just the 2-term **Taylor Expansion** of $f(x)$.)

If we know that value of f at x_0, this gives us a way to approximate the value of f at x near x_0. We do this by starting at $(x_0, f(x_0))$ and moving along the tangent line to approximate the value of the function at x.

Look at $f(x) = \arctan x$. For this applet, click the above link or go to URL http://www.math.hmc.edu/calculus/tutorials/tangent_line/applet3.html. The function being studied in this applet is $f(x) = \arctan x$.

Let's use the tangent approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ to approximate $f(1.04)$:

- Now $f'(x) = \left[\frac{1}{1+x^2}\right]$ so $f'(1) = \left[\frac{1}{1+1^2}\right] = \frac{1}{2}$.

- Let $x_0 = 1$ and $x = 1.04$.
- Then $f(1.04) \approx f(1) + f'(1)(1.04 - 1) \approx \frac{\pi}{4} + \frac{1}{2}(0.04) \approx 0.81$.

How well does this approximate arctan(1.04)?

- Display the tangent through $\left(1, \frac{\pi}{4}\right)$.
- Zoom in on the point to see geometrically how close together the curve and the tangent line are at $x = 1.04$.

Key Concepts

- For the curve $y = f(x)$, the slope of the tangent line at a point (x_0, y_0) on the curve is $f'(x_0)$. The equation of the tangent line is given by

$$y - y_0 = f'(x_0)(x - x_0).$$

- For x close to x_0, the value of $f(x)$ may be approximated by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]