Derivation of the Tangent Plane Equation

Let \((x_0, y_0, z_0)\) be any point on the surface \(z = f(x, y)\) such that \(f(x, y)\) is differentiable at \((x_0, y_0)\). We well show that the tangent plane is normal to the vector \(\mathbf{n} = (f_x(x_0, y_0), f_y(x_0, y_0), -1)\).

Consider any smooth curve \(C\) on the surface that passes through \((x_0, y_0, z_0)\). Parametrize the curve as

\[
\begin{align*}
x &= x(s) \\
y &= y(s) \\
z &= z(s).
\end{align*}
\]

Let \(s = s_0\) satisfy \(x_0 = x(s_0), \quad y_0 = y(s_0), \quad z_0 = z(s_0)\). Then

\[
z(s) = f(x(s), y(s))
\]

for all \(s\). Using the Chain Rule to differentiate both sides of this equation,

\[
\frac{dz}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds}.
\]

Thus,

\[
\frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds} - \frac{dz}{ds} = 0.
\]

Switching notation and writing the equation in vector form,

\[
(f_x(x, y), f_y(x, y), -1) \cdot (x'(s), y'(s), z'(s)) = 0.
\]

At \(s = s_0\),

\[
(f_x(x_0, y_0), f_y(x_0, y_0), -1) \cdot (x'(s_0), y'(s_0), z'(s_0)) = 0.
\]

But \((x'(s_0), y'(s_0), z'(s_0))\) is the tangent vector to the curve \(C\) at \((x_0, y_0, z_0)\). Thus, we have that the tangent vector to any smooth curve \(C\) on the surface that passes through \((x_0, y_0, z_0)\) is normal to the vector \(\mathbf{n} = (f_x(x_0, y_0), f_y(x_0, y_0), -1)\) and so is given by the equation

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.
\]

(This proof is taken from *Calculus*, by Howard Anton.)