Harvey Mudd College Math Tutorial:

Taylor's Theorem

Suppose we're working with a function $f(x)$ that is continuous and has $n+1$ continuous derivatives on an interval about $x = 0$. We can approximate f near 0 by a polynomial $P_n(x)$ of degree n:

- For $n = 0$, the best constant approximation near 0 is

  $$P_0(x) = f(0)$$

  which matches f at 0.

- For $n = 1$, the best linear approximation near 0 is

  $$P_1(x) = f(0) + f'(0)x.$$

  Note that P_1 matches f at 0 and P_1' matches f' at 0.

- For $n = 2$, the best quadratic approximation near 0 is

  $$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2.$$

  Note that P_2, P_2', and P_2'' match f, f', and f'', respectively, at 0.

Continuing this process,

$$P_n(x) = f(0) + f'(0)x + \frac{f''(x)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n.$$

This is the **Taylor polynomial of degree n about 0** (also called the **Maclaurin series of degree n**). More generally, if f has $n+1$ continuous derivatives at $x = a$, the **Taylor series of degree n about a** is

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This formula approximates $f(x)$ near a. Taylor's Theorem gives bounds for the error in this approximation:

Taylor's Theorem

Suppose f has $n+1$ continuous derivatives on an open interval containing a. Then for each x in the interval,

$$f(x) = \left[\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] + R_{n+1}(x)$$

where the error term $R_{n+1}(x)$ satisfies $R_{n+1}(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

This form for the error $R_{n+1}(x)$, derived in 1797 by Joseph Lagrange, is called the Lagrange formula for the remainder. The *infinite* Taylor series converges to f,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

if and only if $\lim_{n\to\infty} R_n(x) = 0$.

Examples of Taylor Series about 0

1. For $f(x) = e^x$,

$$f^{(k)}(x) = e^x \quad \Longrightarrow \quad f^{(k)}(0) = 1.$$

   So

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \end{aligned}$$

   which converges for all x since $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \dfrac{e^c x^{(n+1)}}{(n+1)!} = 0$ for all c between 0 and x.

2. For $f(x) = \ln(1+x)$,

$$\left.\begin{array}{l} f(x) = \ln(1+x) \\ f'(x) = \frac{1}{1+x} \\ f''(x) = \frac{-1}{(1+x)^2} \\ f'''(x) = \frac{2}{(1+x)^3} \\ f^{(4)}(x) = \frac{-3\cdot 2}{(1+x)^4} \\ \vdots \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} f(0) = 0 \\ f'(0) = 1 \\ f''(0) = -1 \\ f'''(0) = 2 \\ f^{(4)}(x) = -6 \\ \vdots \end{array}\right.$$

So

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \end{aligned}$$

which converges only for $-1 < x \leq 1$.

The Taylor Series in $(x-a)$ is the *unique* power series in $(x-a)$ converging to $f(x)$ on an interval containing a. For this reason,

- By Example 1,
$$e^{-2x} = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \dots$$
where we have substituted $-2x$ for x.

- By Example 2, since $\frac{d}{dx}[\ln(1+x)] = \frac{1}{1+x}$, we can differentiate the Taylor series for $\ln(1+x)$ to obtain
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots.$$
Substituting $-x$ for x,
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots.$$

In the Exploration, compare the graphs of various functions with their first through fourth degree Taylor polynomials.

Exploration

Key Concepts

Taylor's Theorem

Suppose f has $n+1$ continuous derivatives on an open interval containing a. Then for each x in the interval,

$$f(x) = \left[\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right] + R_{n+1}(x)$$

where the error term $R_{n+1}(x)$ satisfies $R_{n+1}(x) = \left[\frac{f^{(n+1)}(c)}{(n+1)!} \right] (x-a)^{n+1}$ for some c between a and x.

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]