Harvey Mudd College Math Tutorial:

Special Trigonometric Integrals

In the study of Fourier Series, you will find that every continuous function f on an interval $[-L, L]$ can be expressed on that interval as an infinite series of sines and cosines. For example, if the interval is $[-\pi, \pi]$,

$$f(x) = A_0 + \sum_{k=1}^{\infty} [A_k \cos(kx) + B_k \sin(kx)]$$

where the constants are given by integrals involving f.

The theory of Fourier series relies on the fact that the functions

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \ldots, \quad \cos nx, \quad \sin nx, \quad \ldots$$

form an **orthogonal set**:

The integral of the product of any 2 of these functions over $[-\pi, \pi]$ is 0.

Here, we will verify this fact.

We will use the following trigonometric identities:

$$\begin{aligned} \sin A \sin B &= \frac{1}{2}[\cos(A-B) - \cos(A+B)] \\ \cos A \cos B &= \frac{1}{2}[\cos(A-B) + \cos(A+B)] \\ \sin A \cos B &= \frac{1}{2}[\sin(A-B) + \sin(A+B)]. \end{aligned}$$

We have six general integrals to evaluate to prove the orthogonality of the set $\{1, \ \cos x, \ \sin x, \ldots\}$. In each of the following, we assume m and n are distinct positive integers.

1. $\displaystyle \int_{-\pi}^{\pi} 1 \cdot \cos(nx)\, dx = \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} = 0.$

2. $\displaystyle \int_{-\pi}^{\pi} 1 \cdot \sin(nx)\, dx = -\frac{1}{n} \cos(nx) \Big|_{-\pi}^{\pi} = 0.$

3. $\displaystyle \int_{-\pi}^{\pi} \sin(nx)\cos(nx)\, dx = \frac{\sin^2(nx)}{2n} \Big|_{-\pi}^{\pi} = 0.$

4.

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]\,dx \\ &= \left(\frac{\sin[(m-n)x]}{2(m-n)} - \frac{\sin[(m+n)x]}{2(m+n)}\right)\Bigg|_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

5.

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx &= \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]\,dx \\ &= \left(\frac{\sin[(m-n)x]}{2(m-n)} + \frac{\sin[(m+n)x]}{2(m+n)}\right)\Bigg|_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

6.

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx &= \int_{-\pi}^{\pi} \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]\,dx \\ &= \left(-\frac{\cos[(m-n)x]}{2(m-n)} - \frac{\cos[(m+n)x]}{2(m+n)}\right)\Bigg|_{-\pi}^{\pi} \\ &= 0.\end{aligned}$$

We have now shown that $\{1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x, \ldots\}$ is indeed an orthogonal set of functions!

In the following Exploration, graph functions $\sin(mx)\sin(nx)$, $\sin(mx)\cos(nx)$, and $\cos(mx)\cos(nx)$ for various values of m and n and observe the interesting curves that result.

Exploration

Key Concepts

The theory of Fourier series relies on the fact that the functions
$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots,\ \cos nx,\ \sin nx, \ldots$ form an *orthogonal set*:

The integral of the product of any 2 of these functions over $[-\pi, \pi]$ is 0.

[I'm ready to take the quiz.] [I need to review more.]
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