Harvey Mudd College Math Tutorial:
Special Trigonometric Integrals

In the study of Fourier Series, you will find that every continuous function $f$ on an interval $[-L, L]$ can be expressed on that interval as an infinite series of sines and cosines. For example, if the interval is $[-\pi, \pi],
\[ f(x) = A_0 + \sum_{k=1}^{\infty} [A_k \cos(kx) + B_k \sin(kx)] \]
where the constants are given by integrals involving $f$. The theory of Fourier series relies on the fact that the functions
\[ 1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots \]
form an orthogonal set:

*The integral of the product of any 2 of these functions over $[-\pi, \pi]$ is 0.*

Here, we will verify this fact.

We will use the following trigonometric identities:
\[
\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]
\]
\[
\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]
\]
\[
\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)].
\]

We have six general integrals to evaluate to prove the orthogonality of the set \{1, $\cos x$, $\sin x$, $\ldots$\}. In each of the following, we assume $m$ and $n$ are distinct positive integers.

1. \[
\int_{-\pi}^{\pi} 1 \cdot \cos(nx) \, dx = \frac{1}{n} \sin(nx) \bigg|_{-\pi}^{\pi} = 0.
\]
2. \[
\int_{-\pi}^{\pi} 1 \cdot \sin(nx) \, dx = -\frac{1}{n} \cos(nx) \bigg|_{-\pi}^{\pi} = 0.
\]
3. \[
\int_{-\pi}^{\pi} \sin(nx) \cos(nx) \, dx = \frac{\sin^2(nx)}{2n} \bigg|_{-\pi}^{\pi} = 0.
\]
4. 
\[ \int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos((m-n)x) - \cos((m+n)x)] \, dx \]
\[ = \left. \left( \frac{\sin((m-n)x)}{2(m-n)} - \frac{\sin((m+n)x)}{2(m+n)} \right) \right|_{-\pi}^{\pi} \]
\[ = 0. \]

5. 
\[ \int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = \int_{-\pi}^{\pi} \frac{1}{2} [\cos((m-n)x) + \cos((m+n)x)] \, dx \]
\[ = \left. \left( \frac{\sin((m-n)x)}{2(m-n)} + \frac{\sin((m+n)x)}{2(m+n)} \right) \right|_{-\pi}^{\pi} \]
\[ = 0. \]

6. 
\[ \int_{-\pi}^{\pi} \sin(mx) \cos(nx) \, dx = \int_{-\pi}^{\pi} \frac{1}{2} [\sin((m-n)x) + \sin((m+n)x)] \, dx \]
\[ = \left. \left( -\cos((m-n)x) + \cos((m+n)x) \right) \right|_{-\pi}^{\pi} \]
\[ = 0. \]

We have now shown that \{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\} is indeed an orthogonal set of functions!

In the following Exploration, graph functions \(\sin(mx) \sin(nx), \sin(mx) \cos(nx),\) and \(\cos(mx) \cos(nx)\) for various values of \(m\) and \(n\) and observe the interesting curves that result.

**Exploration**

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**Key Concepts**

The theory of Fourier series relies on the fact that the functions
\(1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots\) form an orthogonal set:

The integral of the product of any 2 of these functions over \([-\pi, \pi]\) is 0.