Harvey Mudd College Math Tutorial:

Trigonometric Substitutions

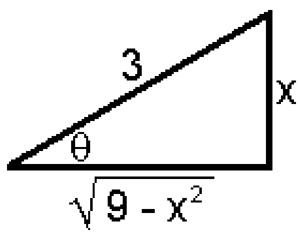
Consider the integral

$$\int \frac{dx}{\sqrt{9-x^2}}.$$

At first glance, we might try the substitution $u = 9 - x^2$, but this will actually make the integral even more complicated!

Let's try a different approach:

The radical $\sqrt{9-x^2}$ represents the length of the base of a right triangle with height x and hypotenuse of length 3:



For this triangle, $\sin\theta = \dfrac{x}{3}$, suggesting the substitution $x = 3\sin\theta$. Then $\theta = \arcsin\left(\dfrac{x}{3}\right)$, where we specify $-\pi/2 \le \theta \le \pi/2$. Note that $dx = 3\cos\theta\, d\theta$ and that $\sqrt{9-x^2} = 3\cos\theta$.

With this change of variables,

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{3\cos\theta\, d\theta}{3\cos\theta} = \int d\theta = \theta + C = \arcsin\left(\frac{x}{3}\right) + C.$$

Caution!

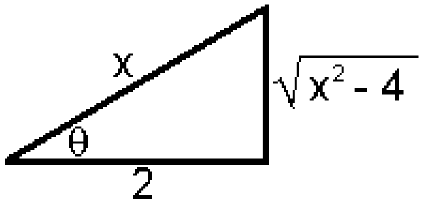
- The sketch of the triangle is very useful for determining what substitution should be made. Note, though, that the sketch only has meaning for $x > 0$ and $\theta > 0$.
- It is important to be careful about how the angle θ is defined. With the restrictions on θ mentioned in the examples here, we avoid sign difficulties even when $x < 0$.

There are two other trigonometric substitutions useful in integrals with different forms:

Example
Let's evaluate

$$\int \frac{dx}{x^2\sqrt{x^2-4}}.$$

The radical $\sqrt{x^2-4}$ suggests a triangle with hypotenuse of length x and base of length 2:



For this triangle, $\sec\theta = \dfrac{x}{2}$, we will try the substitution $x = 2\sec\theta$. Then $\theta = \sec^{-1}\left(\dfrac{x}{2}\right)$, where we specify $0 \leq \theta < \pi/2$ or $\pi \leq \theta < 3\pi/2$. Note that $dx = 2\sec\theta\tan\theta\,d\theta$ and that $\sqrt{x^2-4} = 2\tan\theta$.

Then

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta}{(2\sec\theta)^2(2\tan\theta)}\,d\theta = \int \frac{1}{4}\cos\theta\,d\theta = \frac{1}{4}\sin\theta + C.$$

But we see from the sketch that $\sin\theta = \dfrac{\sqrt{x^2-4}}{x}$, so

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{\sqrt{x^2-4}}{4x} + C.$$

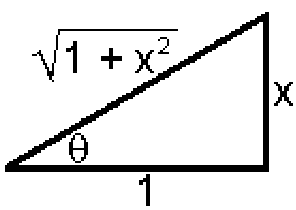
We may also use a trigonometric substitution to evaluate a *definite* integral, as long as care is taken in working with the limits of integration:

Example

We will evaluate

$$\int_{-1}^{1} \frac{dx}{(1+x^2)^2}.$$

The factor $(1+x^2)$ suggests a triangle with base of length 1 and height x:



For this triangle, $\tan\theta = x$, so we will try the substitution $x = \tan\theta$. Then $\theta = \tan^{-1}(x)$, where we specify $-\pi/2 < \theta < \pi/2$. Here, $dx = \sec^2\theta\,d\theta$. Also, $\sqrt{1+x^2} = \sec\theta$ so $(1+x^2)^2 = \sec^4\theta$.

Then

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{(1+x^2)^2} &= \int_{-\pi/4}^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \cos^2\theta\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{2}(1+\cos 2\theta)\,d\theta \\
&= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_{-\pi/4}^{\pi/4} \\
&= = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{1}{2}\left(-\frac{\pi}{4} - \frac{1}{2}\right) \\
&= \frac{\pi}{4} + \frac{1}{2}.
\end{aligned}$$

There is often more than one way to solve a particular integral. A trigonometric substitution will not always be necessary, even when the types of factors seen above appear. With practice, you will gain insight into what kind of substitution will work best for a particular integral.

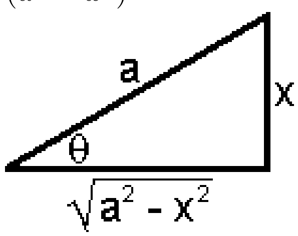
Key Concepts

Trigonometric substitutions are often useful for integrals containing factors of the form

$$(a^2 - x^2)^n, \qquad (x^2 + a^2)^n, \qquad \text{or} \qquad (x^2 - a^2)^n.$$

The exact substitution used depends on the form of the integral:

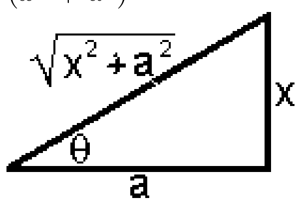
$(a^2 - x^2)^n$



$x = a \sin\theta$
$dx = a \cos\theta$

$\sqrt{a^2 - x^2} = a \cos\theta$

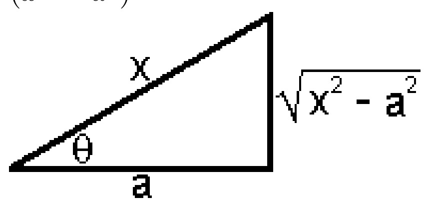
$(x^2 + a^2)^n$



$x = a \tan\theta$
$dx = a \sec^2\theta$

$\sqrt{x^2 + a^2} = a \sec\theta$

$(x^2 - a^2)^n$



$x = a \sec\theta$
$dx = a \sec\theta \tan\theta$

$\sqrt{x^2 - a^2} = a \tan\theta$

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]