Harvey Mudd College Math Tutorial:

Volume

Many three-dimensional solids can be generated by revolving a curve about the x-axis or y-axis. For example, if we revolve the semi-circle given by $f(x) = \sqrt{r^2 - x^2}$ about the x-axis, we obtain a sphere of radius r. We can derive the familiar formula for the volume of this sphere.

Finding the Volume of a Sphere

Consider a cross-section of the sphere as shown. It is a circle with radius $f(x)$ and area $\pi[f(x)]^2$. Informally speaking, if we "slice" the sphere vertically into discs, each disc having infinitesimal thickness dx, the volume of each disc is approximately $\pi[f(x)]^2\,dx$. If we "add up" the volumes of the discs, we will get the volume of the sphere:

$$\begin{aligned}
V &= \int_{-r}^{r} \pi[f(x)]^2\,dx \\
&= \int_{-r}^{r} \pi(r^2 - x^2)\,dx \\
&= \pi\left(r^2x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} \\
&= \pi\left(\frac{2}{3}r^3\right) - \pi\left(-\frac{2}{3}r^3\right) \\
&= \frac{4}{3}\pi r^3, \quad \text{as expected.}
\end{aligned}$$

This is called the **Method of Discs**. In general, suppose $y = f(x)$ is nonnegative and continuous on $[a, b]$. If the region bounded above by the graph of f, below by the x-axis, and on the sides by $x = a$ and $x = b$ is revolved about the x-axis, the volume V of the generated solid is given by

$$V = \int_{b}^{a} \pi[f(x)]^2\,dx.$$

We can also obtain solids by revolving curves about the y-axis.

Revolving a Region about the y-axis

If we revolve the region enclosed by $y = x^2$ and $y = 2x$, $0 \leq x \leq 2$, about the y-axis, we generate a three-dimensional solid.

Let's find the volume of this solid. If we "slice" the solid horizontally, each slice is a "washer." The outer radius is $\sqrt{y}$ (since $y = x^2 \to x = \sqrt{y}$), the inner radius is $y/2$ (since $y = 2x \to x = y/2$), and the thickness is dy. The volume of each washer is therefore

$$[\pi(\sqrt{y})^2 - \pi(y/2)^2]\,dy.$$

Then the volume of the entire solid is given by

$$\begin{aligned}
\int_0^4 [\pi(\sqrt{y})^2 - \pi(y/2)^2]\, dy &= \int_0^4 \pi \left[y - \frac{y^2}{4} \right] dy \\
&= \pi \left[\frac{y^2}{2} - \frac{y^3}{12} \right] \Bigg|_0^4 \\
&= \pi \left(8 - \frac{16}{3} \right) - \pi\,(0-0) \\
&= \frac{8\pi}{3}.
\end{aligned}$$

This generalization of the Method of Discs is called the **Method of Washers**. As we have seen, these methods may be used when a region is revolved about either axis.

Suppose $y = f(x)$ and $y = g(x)$ are continuous and nonnegative on $[a, b]$ with $g(x) \leq f(x)$ for all $x \in [a, b]$. If the region bounded above by f, below by g, and on the sides by $x = a$ and $x = b$ is revolved about the x axis, the volume of the solid generated is $V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\, dx$.

Suppose $x = F(y)$ and $x = G(y)$ are continuous and nonnegative on $[c, d]$ with $G(y) \leq F(y)$ for all $y \in [c, d]$. If the region bounded on the right by F, on the left by G, and on the top and bottom by $y = d$ and $y = c$ is revolved about the y axis, the volume of the solid generated is $V = \int_c^d \pi([F(y)]^2 - [G(y)]^2)\, dy$.

We could have taken a different approach in the previous example:

Another Method

Look again at the volume of the solid generated by revolving the region enclosed by $y = 2x$, $y = x^2$, $0 \leq x \leq 2$ about the y-axis. This time, we will view the solid as being composed of a collection of concentric cylindrical shells of radius x, height $2x - x^2$, and infinitesimal thickness dx. The volume of each shell is approximately given by the lateral surface area ($= 2\pi \cdot \text{radius} \cdot \text{height}$) multiplied by the thickness:

$$2\pi x[2x - x^2]\, dx.$$

"Adding up" the volumes of the cylindrical shells,

$$\begin{aligned}
V &= \int_0^2 2\pi x[2x - x^2]\, dx \\
&= \int_0^2 2\pi[2x^2 - x^3]\, dx \\
&= \left(\frac{4}{3}\pi x^3 - \frac{1}{2}\pi x^4 \right) \Bigg|_0^2 \\
&= \left(\frac{32}{3}\pi - 8\pi \right) - (0-0)
\end{aligned}$$

$$= \frac{8\pi}{3}, \quad \text{as found earlier.}$$

This is called the **Method of Cylindrical Shells**. Suppose $f(x)$, $g(x)$, $F(y)$, $G(y)$ satisfy all the requirements given earlier. Then, for a region revolved about the y-axis,

$$V = \int_a^b 2\pi x f(x)\, dx \qquad \text{or} \qquad V = \int_a^b 2\pi x[f(x) - g(x)]\, dx.$$

For a region revolved about the x-axis,

$$V = \int_c^d 2\pi y F(y)\, dy \qquad \text{or} \qquad V = \int_c^d 2\pi y[F(y) - G(y)]\, dy.$$

Notes

- In the disc and washer methods, you integrate with respect to the *same* variable as the axis about which you revolved the region.
- In the method of cylindrical shells, you integrate with respect to the *other* variable.

Computing volumes using these methods takes some practice. With experience, you will be better able to visualize the solids and determine which method to apply.

Key Concepts

Method of Washers:

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\, dx \qquad \text{or} \qquad V = \int_c^d \pi([F(y)]^2 - [G(y)]^2)\, dy.$$

Method of Cylindrical Shells:

$$V = \int_a^b 2\pi x f(x)\, dx \qquad \text{or} \qquad V = \int_a^b 2\pi x[f(x) - g(x)]\, dx.$$
$$V = \int_c^d 2\pi y F(y)\, dy \qquad \text{or} \qquad V = \int_c^d 2\pi y[F(y) - G(y)]\, dy.$$

[I'm ready to take the quiz.] [I need to review more.]
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