Volume

Many three-dimensional solids can be generated by revolving a curve about the $x$-axis or $y$-axis. For example, if we revolve the semi-circle given by $f(x) = \sqrt{r^2 - x^2}$ about the $x$-axis, we obtain a sphere of radius $r$. We can derive the familiar formula for the volume of this sphere.

**Finding the Volume of a Sphere**

Consider a cross-section of the sphere as shown. It is a circle with radius $f(x)$ and area $\pi [f(x)]^2$. Informally speaking, if we “slice” the sphere vertically into discs, each disc having infinitesimal thickness $dx$, the volume of each disc is approximately $\pi [f(x)]^2 dx$. If we “add up” the volumes of the discs, we will get the volume of the sphere:

$$V = \int_{-r}^{r} \pi [f(x)]^2 dx$$
$$= \int_{-r}^{r} \pi (r^2 - x^2) dx$$
$$= \pi \left( r^2x - \frac{x^3}{3} \right) \bigg|_{-r}^{r}$$
$$= \pi \left( 2 \frac{2}{3}r^3 \right) - \pi \left( -2 \frac{2}{3}r^3 \right)$$
$$= 4 \frac{2}{3} \pi r^3, \quad \text{as expected.}$$

This is called the **Method of Discs**. In general, suppose $y = f(x)$ is nonnegative and continuous on $[a, b]$. If the region bounded above by the graph of $f$, below by the $x$-axis, and on the sides by $x = a$ and $x = b$ is revolved about the $x$-axis, the volume $V$ of the generated solid is given by

$$V = \int_{a}^{b} \pi [f(x)]^2 dx.$$

We can also obtain solids by revolving curves about the $y$-axis.

**Revolving a Region about the $y$-axis**

If we revolve the region enclosed by $y = x^2$ and $y = 2x$, $0 \leq x \leq 2$, about the $y$-axis, we generate a three-dimensional solid.

Let’s find the volume of this solid. If we “slice” the solid horizontally, each slice is a “washer.” The outer radius is $\sqrt{y}$ (since $y = x^2 \to x = \sqrt{y}$), the inner radius is $y/2$ (since $y = 2x \to x = y/2$), and the thickness is $dy$. The volume of each washer is therefore

$$[\pi(\sqrt{y})^2 - \pi \left( \frac{y}{2} \right)^2] dy.$$
Then the volume of the entire solid is given by
\[
\int_0^4 \left[ \pi (\sqrt{y})^2 - \pi (y/2)^2 \right] dy = \int_0^4 \pi \left[ y - \frac{y^2}{4} \right] dy
\]
\[
= \pi \left[ \frac{y^2}{2} - \frac{y^3}{12} \right]_0^4
\]
\[
= \pi \left( 8 - \frac{16}{3} \right) - \pi (0 - 0)
\]
\[
= \frac{8\pi}{3}.
\]
This generalization of the Method of Discs is called the **Method of Washers**. As we have seen, these methods may be used when a region is revolved about either axis.

Suppose \( y = f(x) \) and \( y = g(x) \) are continuous and nonnegative on \([a, b]\) with \( g(x) \leq f(x) \) for all \( x \in [a, b] \). If the region bounded above by \( f \), below by \( g \), and on the sides by \( x = a \) and \( x = b \) is revolved about the \( x \) axis, the volume of the solid generated is
\[
V = \int_a^b \pi ([f(x)]^2 - [g(x)]^2) \, dx.
\]

Suppose \( x = F(y) \) and \( x = G(y) \) are continuous and nonnegative on \([c, d]\) with \( G(y) \leq F(y) \) for all \( y \in [c, d] \). If the region bounded on the right by \( F \), on the left by \( G \), and on the top and bottom by \( y = d \) and \( y = c \) is revolved about the \( y \) axis, the volume of the solid generated is
\[
V = \int_c^d \pi ([F(y)]^2 - [G(y)]^2) \, dy.
\]

We could have taken a different approach in the previous example:

**Another Method**

Look again at the volume of the solid generated by revolving the region enclosed by \( y = 2x \), \( y = x^2 \), \( 0 \leq x \leq 2 \) about the \( y \)-axis. This time, we will view the solid as being composed of a collection of concentric cylindrical shells of radius \( x \), height \( 2x - x^2 \), and infinitesimal thickness \( dx \). The volume of each shell is approximately given by the lateral surface area \((= 2\pi \cdot \text{radius} \cdot \text{height})\) multiplied by the thickness:
\[
2\pi x [2x - x^2] \, dx.
\]

“Adding up” the volumes of the cylindrical shells,
\[
V = \int_0^2 2\pi x [2x - x^2] \, dx
\]
\[
= \int_0^2 2\pi [2x^2 - x^3] \, dx
\]
\[
= \left( \frac{4}{3} \pi x^3 - \frac{1}{2} \pi x^4 \right)_0^2
\]
\[
= \left( \frac{32}{3} \pi - 8\pi \right) - (0 - 0)
\]
\[ = \frac{8\pi}{3}, \text{ as found earlier.} \]

This is called the **Method of Cylindrical Shells**. Suppose \( f(x), g(x), F(y), G(y) \) satisfy all the requirements given earlier. Then, for a region revolved about the \( y \)-axis,

\[ V = \int_{a}^{b} 2\pi x f(x) \, dx \quad \text{or} \quad V = \int_{a}^{b} 2\pi x [f(x) - g(x)] \, dx. \]

For a region revolved about the \( x \)-axis,

\[ V = \int_{c}^{d} 2\pi y F(y) \, dy \quad \text{or} \quad V = \int_{c}^{d} 2\pi y [F(y) - G(y)] \, dy. \]

**Notes**

- In the disc and washer methods, you integrate with respect to the *same* variable as the axis about which you revolved the region.

- In the method of cylindrical shells, you integrate with respect to the *other* variable.

Computing volumes using these methods takes some practice. With experience, you will be better able to visualize the solids and determine which method to apply.

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**Key Concepts**

**Method of Washers:**

\[ V = \int_{a}^{b} \pi \left([f(x)]^2 - [g(x)]^2\right) \, dx \quad \text{or} \quad V = \int_{c}^{d} \pi \left([F(y)]^2 - [G(y)]^2\right) \, dy. \]

**Method of Cylindrical Shells:**

\[ V = \int_{a}^{b} 2\pi x f(x) \, dx \quad \text{or} \quad V = \int_{a}^{b} 2\pi x [f(x) - g(x)] \, dx. \]

\[ V = \int_{c}^{d} 2\pi y F(y) \, dy \quad \text{or} \quad V = \int_{c}^{d} 2\pi y [F(y) - G(y)] \, dy. \]