Orthogonal Functions, $L^2$ Convergence and Fourier Series

**Due date:** Thursday, March 7th in class.

For these problems, you may use MAPLE when appropriate.

**F1: Gram-Schmidt orthogonalization.** We know from elementary linear algebra that any set of linearly independent vectors may be turned into a set of orthogonal vectors by the Gram-Schmidt orthogonalization process. The same process works for functions defined on an interval $a < x < b$ (a mathematician would say on $L^2[a,b]$). Let $\{f_n\}$ be a linearly independent set of functions. Define an inner-product
\[
\langle p(x), q(x) \rangle = \int_a^b p(x)q(x) \, dx.
\]
The sequence of functions $g_n$ is defined inductively by
\[
g_1 = f_1, \quad g_2 = f_2 - \frac{\langle f_2, g_1 \rangle}{\|g_1\|^2} g_1, \quad g_3 = f_3 - \frac{\langle f_3, g_1 \rangle}{\|g_1\|^2} g_1 - \frac{\langle f_3, g_2 \rangle}{\|g_2\|^2} g_2, \ldots.
\]
where $\|g\|^2 = \langle g, g \rangle$. Use induction to show that $g_n$ is an orthogonal sequence.

**F2: Approximating a function by Legendre Polynomials.** This is a great problem for you to hone your MAPLE skills on.

(a) The functions $\{1, x, x^2, x^3, \ldots\}$, are linearly independent on the interval $x \in [-1, 1]$. Use the previous problem and the functions $\{1, x, x^2, x^3\}$ to generate a sequence of four orthogonal polynomials on $x \in [-1, 1]$. Denote the polynomials by $P_0(x), P_1(x), P_2(x), P_3(x)$ (they are called the Legendre Polynomials).

(b) Find an approximation of $e^x$ on $x \in [-1, 1]$ of the form
\[
e^x \approx c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x)
\]
that is best in the root-mean-square sense, and graph $e^x$ and the approximation on a set of coordinate axes.

(c) For your approximation what is the maximum of the pointwise error on $[-1, 1]$? What is the $L^2$ error?

**Editorial Note:** The pointwise error of an approximation $F^*(x)$ to a function $F(x)$ is $E(x) = |F(x) - F^*(x)|$. The root-mean-square or $L^2$ error here is
\[
\|E(x)\| = \sqrt{\int_{-1}^{1} [E(x)]^2 \, dx}
\]

(please turn over)
**F3:** Consider a periodic function, \( g(t) \), with period \( 2\pi \) and 
\[
g(t) = t^2 \quad -\pi < t < \pi
\]

(a) Find a Fourier cosine series for \( g(t) \). Check your answer by graphing the partial sums with MAPLE.

(b) Differentiate the cosine series and the function to obtain a sine series for 
\[
f(t) = \frac{1}{2} \frac{dg}{dt} = t \quad -\pi < t < \pi
\]
Check your answer by graphing the partial sums with MAPLE. What happens at \( t = \pm \pi \)?

(c) Show by consider \( g(0) \) that 
\[
\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.
\]
What do you need to assume for this identity to be true?

(d) Prove the two identities
\[
\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}
\]
by applying Parseval’s Theorem to \( f(t) \) and \( g(t) \) respectively. What have you assumed here and how does it differ from part (c)?

**F4:** Consider a function, \( h(x) \), defined for \( -\pi < x \leq \pi \)
\[
h(x) = \begin{cases} 
2x & 0 \leq x \leq \pi \\
0 & -\pi < x \leq 0
\end{cases}
\]

(a) Draw the \( 2\pi \)-periodic extension of \( h(x) \).

(b) Compute the Fourier expansion for \( h(x) \),
\[
h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) .
\]
Graph some partial sums and show that it converges to the periodic extension of \( h(x) \). What happens at \( \pi \)?
(c) Note that the even part of $h(x)$ is given by

$$E(x) = \frac{1}{2} [h(x) + h(-x)] = |x| \quad -\pi \leq x \leq \pi.$$ 

Use this identity to compute the series for $E(x) = |x|$ on this interval. Show the series reduces to a cosine series (why is this true?). You can verify this result with the calculation we did in class.

(d) Note that the odd part of $h(x)$ is given by

$$O(x) = \frac{1}{2} [h(x) - h(-x)] = x \quad -\pi < x < \pi.$$ 

Use this identity to compute the series for $O(x) = x$ on this interval. Show the series reduces to a sine series (why is this true?). You can verify this result with H3(b).