LECTURE 20: CONSEQUENCES OF THE FUNDAMENTAL
THEOREM OF LINE INTEGRALS

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Last time we were introduced to the fundamental theorem of line integrals. We saw in particular that conservative vector fields have path independent line integrals. Let’s recall the FTLI and prove it.

**Theorem 1 (FTLI).** Let $F = \nabla f$ be a conservative vector field along a smooth curve $C$ from $A$ to $B$. Then

$$\oint_C F \cdot d\mathbf{r} = f(B) - f(A).$$

**Proof.** Let $C$ be parametrized by the differentiable path $\mathbf{r}(t)$ so that

$$C = \{\mathbf{r}(t) : a \leq t \leq b\},$$

and $A = \mathbf{r}(a)$ and $B = \mathbf{r}(b)$. Then we compute

$$\oint_C F \cdot d\mathbf{r} = \int_{t=a}^{t=b} F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

(1)

$$= \int_{t=a}^{t=b} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

(2)

$$= \int_{t=a}^{t=b} \frac{df(\mathbf{r}(t))}{dt} dt$$

(3)

$$= f(B) - f(A).$$

Note that line (1) follows because we are given $F = \nabla f$, equality (2) is a statement of the chain rule, and (3) is the result of the Fundamental Theorem of Calculus. □

**Corollary 2.** If $F$ is a conservative vector field, then

$$\oint_C F \cdot d\mathbf{r} = 0$$

for any smooth closed curve $C$.

**Proof.** Suppose that $F$ is conservative. Then there is a scalar potential $f$ such that $F = \nabla f$. Let $C$ be a closed curve. Then

$$\oint_C F \cdot d\mathbf{r} = \int_{t=a}^{t=b} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$= f(B) - f(A)$$

$$= f(A) - f(A) = 0.$$

The last line follows as $A = \mathbf{r}(a) = \mathbf{r}(b) = B$ because $C$ is a closed curve. □

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Theorem 3. The following are equivalent.

(1) \( F \) is a conservative vector field.
(2) For every closed curve \( C \), \( \oint \limits_C F \cdot d\vec{r} = 0 \).

Proof. That (1) implies (2) is the statement of Corollary 2. We show that (2) implies (1). So, suppose that

\[
\oint \limits_C F \cdot d\vec{r} = 0
\]

for any closed curve \( C \). Then we need to find a potential \( f \) such that

\[
F = \nabla f.
\]

First, we fix the constant term of \( f \) by requiring \( f(0, 0) = 0 \). (We could just as well choose any other constant; this choice makes no difference.) We need to define \( f(x, y) \) in such a way that \( F(x, y) = \nabla(x, y) \). We define \( f(x, y) \) as follows.

\[
f(x, y) = \int \limits_C F \cdot d\vec{r},
\]

where \( C \) is any path from \((0, 0)\) to \((x, y)\).

First, we must show that \( f(x, y) \) is well defined. In other words, suppose that we chose another path \( D \) from \((0, 0)\) to \((x, y)\). Then the path \( C \cup -D \) is a closed path beginning and ending at the origin. Therefore

\[
0 = \oint \limits_{C \cup -D} F \cdot d\vec{r} = \oint \limits_C F \cdot d\vec{r} - \oint \limits_D F \cdot d\vec{r},
\]

and hence

\[
\oint \limits_D F \cdot d\vec{r} = \oint \limits_C F \cdot d\vec{r} = f(x, y).
\]
Thus $f(x, y)$ is indeed well defined and does not depend upon the choice of path from the origin to $(x, y)$.

Now we need to prove that $F = \nabla f$. Let $F(x, y) = (M(x, y), N(x, y))$. Then we must show

$$M = f_x \quad \quad \quad N = f_y$$

To begin, let $C$ be the piecewise linear curve consisting of the line segment from the origin to $(0, y)$ and the line segment from $(0, y)$ to $(x, y)$. Then we have

$$f(x, y) = \int_C F \cdot d\vec{r} = \int_{t=0}^{t=y} N(0, t) dt + \int_{t=0}^{t=x} M(t, y) dt.$$  

Thus, by FTOC,

$$f_x = M(x, y).$$

It’s difficult to see $f_y$ from this description.

So, now consider the curve $D$, which consists of the line segment from the origin to $(x, 0)$ and the line segment from $(x, 0)$ to $(x, y)$. Then we have

$$f(x, y) = \int_D F \cdot d\vec{r} = \int_{t=0}^{t=x} M(t, 0) dt + \int_{t=0}^{t=y} N(x, t) dt.$$  

Thus, again by FTOC,

$$f_y = N(x, y).$$

Therefore $\nabla f = (f_x, f_y) = (M, N) = F$.  

\[ \square \]
**Theorem 4.** Let \( F = (M, N) \) be a vector field of class \( C^2 \). If \( F \) is conservative, then
\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.
\]

**Proof.** Suppose \( F = \nabla f \). Then \( M = f_x \) and \( N = f_y \). Thus
\[
N_x = f_{yx} \quad \quad \quad \quad M_y = f_{xy}.
\]
But
\[
f_{xy} = f_{yx}
\]
because \( F \) is of class \( C^2 \).

How about the converse? The answer is yes, as long as we are inspecting a nice region \( R \). Here, nice means having no holes. A more precise definition is *simply connected*, which essentially means having no holes, but the full study of such regions is outside the scope of this course.

**Theorem 5.** Let \( F(x, y) = (M, N) \) be a vector field on a simply connected region \( R \). If \( N_x = M_y \), then \( F \) is conservative.

In fact, we can generalize this to \( \mathbb{R}^3 \).

**Theorem 6.** On a simply connected region, the vector field \( F \) is conservative if and only if \( \text{curl} \ (F) = 0 \).