World Class Sprints

Believe it or not, mathematical equations can be used to model sprints! A simple model arises when looking at a runner in a sprint. She propels herself forward with a propulsive force created by pumping her legs. But there are also physical limitations on her maximum speed due to air resistance, and due to frictional forces inside her body from moving her limbs. For sprints of up to 291 meters, Keller predicts that the optimal running strategy is for the runner to run at her maximum acceleration throughout the race. How do the sprinter's acceleration, velocity, and distance covered change throughout the race?

Our finished model will involve two parameters, called $A$ and $\tau$. Parameters often arise within mathematical models, and they need to be determined using observed data. The parameters in our model will tell us about the abilities of the runner. Data of sprinters from 1973 gave the values $A = 12.2$ m/sec$^2$ and $\tau = 0.892$ sec. The table below holds actual data (courtesy of Shawn Price of Track and Field News) from the 1993 World Championships in Stuttgart, Germany, for the 100 meter sprint for the top four finishers for both the men and the women.

<table>
<thead>
<tr>
<th>Name</th>
<th>Nation</th>
<th>30 meters</th>
<th>60 meters</th>
<th>80 meters</th>
<th>100 meters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linford Christie</td>
<td>GBR</td>
<td>3.85</td>
<td>6.45</td>
<td>8.15</td>
<td>9.87</td>
</tr>
<tr>
<td>Andre Cason</td>
<td>USA</td>
<td>3.83</td>
<td>6.43</td>
<td>8.15</td>
<td>9.92</td>
</tr>
<tr>
<td>Dennis Mitchell</td>
<td>USA</td>
<td>3.82</td>
<td>6.46</td>
<td>8.22</td>
<td>9.99</td>
</tr>
<tr>
<td>Carl Lewis</td>
<td>USA</td>
<td>3.95</td>
<td>6.59</td>
<td>8.30</td>
<td>10.02</td>
</tr>
<tr>
<td>Gail Devers</td>
<td>USA</td>
<td>4.09</td>
<td>6.95</td>
<td>8.86</td>
<td>10.82</td>
</tr>
<tr>
<td>Merlene Ottey</td>
<td>GHA</td>
<td>4.13</td>
<td>6.98</td>
<td>8.87</td>
<td>10.82</td>
</tr>
<tr>
<td>Gwen Torrance</td>
<td>USA</td>
<td>4.14</td>
<td>7.00</td>
<td>8.92</td>
<td>10.89</td>
</tr>
<tr>
<td>Irina Privalova</td>
<td>RUS</td>
<td>4.09</td>
<td>7.00</td>
<td>8.96</td>
<td>10.96</td>
</tr>
</tbody>
</table>

Question.

1. After building the model, determine average values for $A$ and $\tau$ for both the men and the women using the data from the 1993 World Championships. How do the values for the men and women compare? Based on values of $A$ and $\tau$, have the abilities of runners increased since 1973? (There are a couple of ways to do this.)


Building the Model

Let's use Newton's second law to develop a differential equation for the velocity of the runner. We must balance the propulsive force of her legs with the resistive force. Newton's second law says that force and acceleration are proportionally related, \( F = ma \):

\[
F = ma = \text{propulsive force} - \text{resistive force}
\]

The propulsive force is a function \( P(t) = mA(t) \) which varies only with the runner's acceleration \( A(t) \). There is a maximum acceleration with which the runner can run, call it \( A \). Since for sprints the optimal running strategy is to run at maximum acceleration, we will model the propulsive force as \( P(t) = mA \).

The resistive force is due to air resistance and frictional effects resulting from moving the limbs of the body. Air resistance is negligible compared to the other frictional effects, so let's ignore it. The frictional effects can be modeled linearly with respect to velocity, so the resistive force is \(-kv\), acting opposite to the runner. Putting all our pieces together,

\[
ma = mA - kv
\]

Recalling that \( a = \frac{dv}{dt} \) and setting \( \tau = \frac{m}{k} \), this equation can be written as

\[
\frac{dv}{dt} = A - \frac{v}{\tau}
\]

Questions.
2. Make plots of the runner's acceleration, velocity, and distance traveled as functions of time, using the 1973 parameters \( A = 12.2 \text{ m/sec}^2 \) and \( \tau = 0.892 \text{ sec} \).
3. Determine formulas for the acceleration, velocity, and distance traveled as functions of time.
4. What is the runner's maximum velocity and acceleration? Does the runner's maximum velocity depend on the length of the race? How long does it take for the runner's velocity to reach 90% of its maximum value? For the runner's acceleration to decrease to 10% of its maximum value?
WORLD-CLASS SPRINTS I

J.B. Keller created this model for the sprinter's velocity v in m/sec:

\[ v' = \lambda - v/\tau \quad // \quad v(0) = 0 \]

\[ \tau = 0.7; \lambda = 12.2 \]

*Tau is the ratio of the sprinter's mass to a damping constant, and \lambda is the sprinter's initial acceleration. You can see the effect of the parameters \lambda and \tau on performance by sweeping through various parameter values. Do large values of \tau and \lambda give the best performance? How do you find the time T taken to run the 100 m race? What tau-values ensure \( t < 10 \) sec? */
WORLD-CLASS SPRINTS II

Now let's see how the sprinter's velocity depends on the initial acceleration (using Keller's model):

\[
v' = A - v/\tau \]

\[\tau = 0.9; A = 10\]

/* Let's see how the sprinter's distance, velocity, and acceleration vary with time and the parameter A. Sweep A from 10 to 14 in 9 steps while holding \( \tau \) at 0.9. How do you find and graph the distance covered? Does the runner with \( A = 14 \) and \( \tau = 9 \) run the 100 meters in 10 seconds? Compare with real data. */