The Conformal Center of a Triangle or a Quadrilateral

by

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Abstract

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Every triangle has a unique point, called the conformal center, from which a random (Brownian motion) path is equally likely to first exit the triangle through each of its three sides. We use concepts from complex analysis, including harmonic measure and the Schwarz-Christoffel map, to locate this point. We could not obtain an elementary closed-form expression for the conformal center, but we show some series expressions for its coordinates. These expressions yield some new hypergeometric series identities.

Using Maple in conjunction with a homemade Java program, we numerically evaluated these series expressions and compared the conformal center to the known geometric triangle centers. Although the conformal center does not exactly coincide with any of these other centers, it appears to always lie very close to the Second Morley point. We empirically quantify the distance between these points in two different ways.

In addition to triangles, certain other special polygons and circles also have conformal centers. We discuss how to determine whether such a center exists, and where it will be found.

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Acknowledgments

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Chapter 1

Introduction

When a particle moves according to Brownian motion, its direction changes randomly at every instant. If we release such a particle inside a polygon, it will eventually exit the polygon. For each of the polygon's sides, we can determine the probability that the particle will first exit the polygon through that side. This probability is a function of the particle's starting point. For certain special polygons, there is a unique starting point, which we call the conformal center, from which the particle is equally likely to exit through *any* of the sides of the polygon. We formalize this definition in Chapter 2.

In Chapter 3, we show that this probability function is harmonic. In fact, it is given by the well-studied harmonic measure function. This equivalence allows our problem to be studied in the context of complex analysis. We show some immediate results of this equivalence in Chapter 4.

In Chapter 5, we introduce the Schwarz-Christoffel Transformation, a conformal mapping that maps the upper half of the complex plane to a polygon. Because conformal maps preserve harmonic functions, we can use the Schwarz-Christoffel Transformation to find the conformal center of a polygon. First we find the conformal center in the half-plane, then we map it to the desired polygon using the appropriate Schwarz-Christoffel Transformation. This allows us to determine the existence and uniqueness of the conformal center, and gives an integral expression for the conformal center in a polygon.

This much about the conformal center was known before we began our research. During Summer of 2002, we began analyzing the conformal center in triangles. In Chapter 6, we investigate whether the conformal center of a triangle coincides with any known triangle center. We can rule out many of the known centers by considering limiting cases, such as the half-strip obtained by allowing one vertex of an isosceles triangle to go to infinity. Unfortunately, this is not a reasonable way to compare the conformal center to the more than 1000 known geometric triangle centers.

In Chapter 7, we explain how to reduce the integral expression for the conformal center of a triangle to a quickly-converging series. We show that the conformal center fits the definition of a triangle center, and find a series expression for its triangle function.

Using this series, we wrote a Java program to plot the conformal center for any given triangle (Chapter 8). This program can be used, in conjunction with an online index of triangle centers, to show that the conformal center does not exactly coincide with any known geometric triangle center. It does, however, lie very close to the Second Morley point. This marks the end of the work we did during Summer 2002. During the academic year, 2002-2003, we formulated various descriptions of the closeness of these two points. We present these in Chapter 9.

We also consider the conformal center in polygons other than triangles. In Chapter 10, we establish a criterion for determining whether a general polygon has a conformal center. In Chapter 11, we specifically consider quadrilaterals. We prove that every symmetric quadrilateral has a conformal center, and find a series expression for its coordinates.

In Chapter 12, we consider the conformal center of a circle. We divide its circumference into arc segments and define the conformal center to be a point from which a Brownian path is equally likely to first exit the circle through any of these arcs. When we divide the circumference with exactly three points, the circle has a conformal center that coincides with the First Isodynamic center of the triangle with vertices at these points. We also show a simple criterion for determining whether a circle has a conformal center when exactly four points divide its circumference.

Finally, in Chapter 13, we show an application of this work to evaluating hypergeometric series. In the conclusion, we summarize the paper, give some resources for hypergeometric series, and discuss a generalization of the conformal center.

Chapter 2

Definitions

A Brownian-motion path is obtained by taking infinitesimal steps in random directions. We can approximate this motion as follows. First, choose some step size $s > 0$. Now repeatedly choose a random angle from $[0, 2\pi]$ with uniform probability, and move a distance s in the direction of the angle chosen. The path obtained as we make the step size smaller and smaller ($s \to 0$) is a Brownian path [4].

Consider a Brownian path that starts at a point on the interior of a polygon. Over time, the probability that it has exited the polygon goes to one [4]. Since the path is certain to exit the polygon eventually, let us consider which side it exits through first. If the starting point is near a particular side, the path is more likely to exit through that side than if the starting point is far away. For some starting points, a Brownian path may be equally likely to exit from *any* of the sides of the polygon. Formally, we define the conformal center as follows.

Definition. *Let q be a point on the interior of a polygon P. Then q is the* ***conformal center*** *of P if a Brownian-motion path starting at q is equally likely to first exit P from any of its sides.*

Notice that not every polygon has a conformal center. A square has a conformal center, the center point, but a non-square rectangle does not, as shown in Fig. 2.1. Here the lines L_1 and L_2 divide the rectangle in half. If the Brownian path starts above the line L_1, it will be more likely to first exit through side A than through side C, and vice-versa. Consequently, the conformal center, if it exists, must lie on

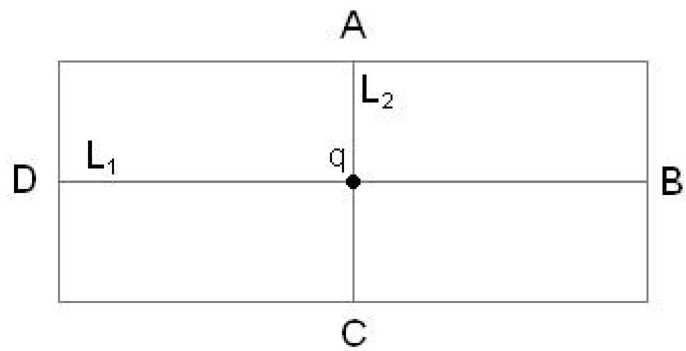

Figure 2.1: A non-square rectangle has no conformal center.

the line L_1. Similarly, it must lie on the line L_2. Thus, only q, which lies on both of these lines, can possibly be a conformal center. However, a path from q is more likely to first exit through A or C than through B or D. Consequently, a path from q is not equally likely to first exit through any of the four sides. We conclude that this rectangle does not have a conformal center. This argument can be formalized using harmonic measure, which we introduce in the next chapter.

Chapter 3

Brownian Motion and Harmonic Measure

In an n-sided polygon, a Brownian path starting at the conformal center has a $1/n$ probability of first exiting through any of the sides. To find this point, it would be helpful to determine the probability that a path from a given point z exits first through a given side S. Fortunately, this probability is completely determined by *harmonic measure*, a concept from complex analysis. This result, shown by Shizuo Kakutani [6], makes our analysis of the conformal center much easier. We can motivate the result as follows.

Suppose a polygon P and one of its sides S is given. Let us place the polygon in the complex plane. For each interior point z, we denote the probability that a Brownian path from z exits first through S as the "probability function," $p(z)$.

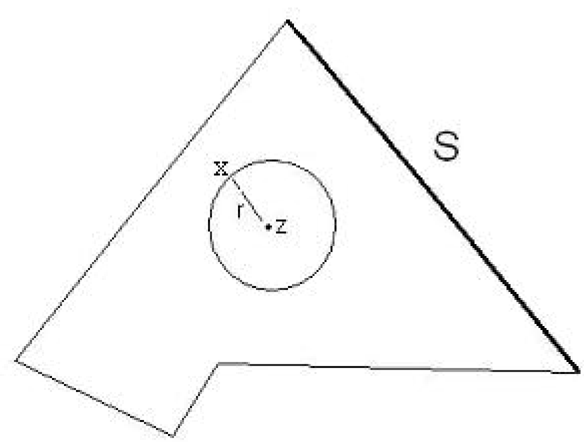

Figure 3.1: The probability $p(z)$ is the average of the probabilities $p(x)$ over the circumference of the circle.

If we draw a circle around the point z, as in Fig. 3.1, then a Brownian path from z is equally likely to first exit this circle through any arc of the circumference of some fixed length. It seems plausible, therefore, that $p(z)$ is the average of $p(x)$ over the points x on the circumference of the circle. Formally, for a circle with radius r,

$$p(z) = \frac{1}{2\pi} \int_0^{2\pi} p(z + re^{i\theta}) d\theta$$

This Mean Value Property of the function p holds for all choices of z and r where the circle generated is entirely contained within the polygon. As the reader may recall from complex analysis, this is a property of harmonic functions [8]. In fact, it is a defining property of harmonic functions [1]. We therefore conclude that p is a harmonic function. Note that any path starting at a point on S necessarily first exits though S, so $p = 1$ on S. Similarly, $p = 0$ at all other boundary points $\partial\Omega \setminus S$.

Fortunately, the function p is actually a better-known function in disguise, namely the *harmonic measure* function. Recall that the Dirichlet problem attempts to find a harmonic function on a complex domain Ω for a given set of values on the boundary $\partial\Omega$. The standard method for solving this problem is the Perron process, which yields a harmonic function with the correct boundary values, except possibly at very small set of boundary points (a set of capacity zero) [1]. In the specific case with boundary values 1 on some subset $E \subset \partial\Omega$ and 0 on $\partial\Omega \setminus E$, the harmonic measure at z, denoted $\omega(z, E, \Omega)$, is the value of this function at z. If we fix Ω and E, then we can treat the harmonic measure as a harmonic function defined at the points $z \in \Omega$.

Kakutani showed that the harmonic function obtained by the Perron process with these boundary values equals the probability that a Brownian motion path first exits $\partial\Omega$ through E on all interior points. That is, the harmonic measure is equal to our Brownian probability p at all interior points of Ω.

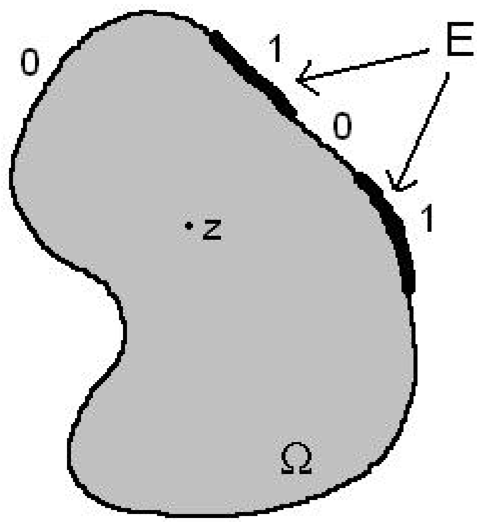

Figure 3.2: The harmonic measure is 1 on E, 0 on $\partial\Omega \setminus E$, and is harmonic on the interior of Ω.

Theorem (Kakutani, 1944). *The probability that a Brownian path from z will first exit Ω through E is the harmonic measure $\omega(z, E, \Omega)$* [6].

This fact allows us to turn the problem of finding the conformal center into a set of Dirichlet problems with boundary values 0 and 1. Fortunately, we often do not need to use the Perron process to find interior values of the harmonic measure. The boundary values of a bounded harmonic function uniquely determine the interior values of that function, so any way we calculate this function will yield the same result at the Perron process. For the domains we are interested in, there are fairly simple ways to calculate such a harmonic function.

To illustrate this technique, suppose we wish to find the conformal center of a given triangle. We place the triangle in the complex plane, and label the interior Ω and the three sides S_1, S_2, and S_3. The conformal center, if it exists, is a point $q \in \Omega$ where all three harmonic measures are equal:

$$\omega(q, S_1, \Omega) = \omega(q, S_2, \Omega) = \omega(q, S_3, \Omega) = \frac{1}{3}$$

For an arbitrary n-sided polygon, the condition satisfied by the conformal center is:

$$\omega(q, S_k, \Omega) = \frac{1}{n}$$

for $k = 1, 2, \ldots, n$. Thus, to find the conformal center, we need to find a point where these conditions hold. Since we have reduced our Brownian motion problem to a set of Dirichlet problems, we will be able to employ powerful tools from complex analysis.

Chapter 4

Some Immediate Results

Even before applying more sophisticated tools from complex analysis, we can already gain an intuitive sense of where a polygon's conformal center is. For example, suppose our polygon is a triangle, and and assume for now that the triangle has a unique conformal center (we will prove this formally later).

In general, we expect the harmonic measure for any side to be close to 1 near that side, and close to 0 near to the other sides. For a long pointy triangle, as in the figure, we therefore expect the harmonic measure of the short side to drop quickly as z moves away from it. The points where this harmonic measure is $1/3$ must therefore be fairly close to the short side. That is, the conformal center should be nearer x than y, as in Fig. 4.1. Similarly, for a flat triangle with one long side, the

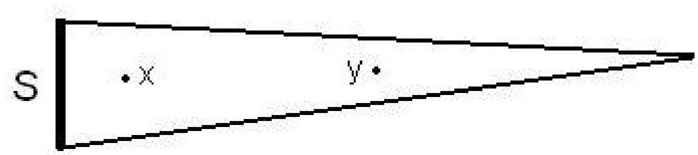

Figure 4.1: The conformal center of a pointy triangle will be close to the short side S, nearer x than y.

conformal center will be close to the other sides, as in Fig. 4.2.

These intuitive arguments could have been made using only the original Brownian-motion definition of the conformal center. We can make more sophisticated arguments using harmonic measure.

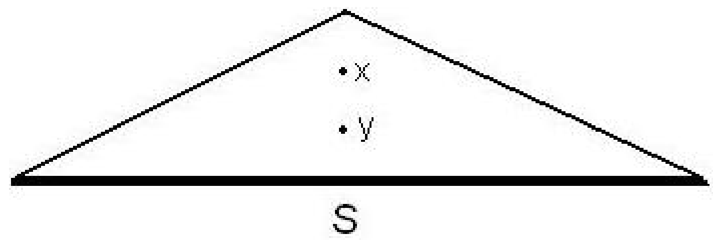

Figure 4.2: In a triangle with one long side S, the conformal center will be closer to the other sides, nearer x than y.

Theorem. *Suppose a line L bisects an angle of a triangle at vertex C, as in Fig. 4.3. If the triangle is not isosceles, then L divides the side opposite C into two segments. The conformal center lies on the side of the smaller segment.*

Proof. Let a triangle be given, with an angle bisected by a line L, as shown in Fig. 4.3-1, and segment A shorter than B. Define ω_1 to be the harmonic function on the Starfleet-symbol shaped region, with boundary values $-1, 0, 1$, as shown in Fig. 4.3-2. By symmetry, $\omega_1 = 0$ on the dashed line, and by the Maximum Principle, $\omega_1 < 0$ on segment A. Next, let ω_2 be the harmonic function with boundary values equal to zero on all parts of the boundary except on A, where $\omega_2 = -\omega_1$, as in Fig. 4.3-3. Notice that $\omega_2 > 0$ along the dashed line. Finally, let $\omega = \omega_1 + \omega_2$. Since ω_1 and ω_2 are harmonic, so is ω. By summing the boundary values for ω_1 and ω_2, we conclude that ω is the solution to the boundary value problem with the values in Fig. 4.3-4. Since $\omega > 0$ along the dashed line, Brownian paths from such points are more likely to first exit by the right side than by the left side. The conformal center must be equally likely to first exit through either of these sides, so it must lie within the shaded region to the left of the dashed line, as desired. □

Another harmonic measure argument shows the barycentric coordinates of the conformal center are bounded above by the value, $2/3$. Recall that the barycentric

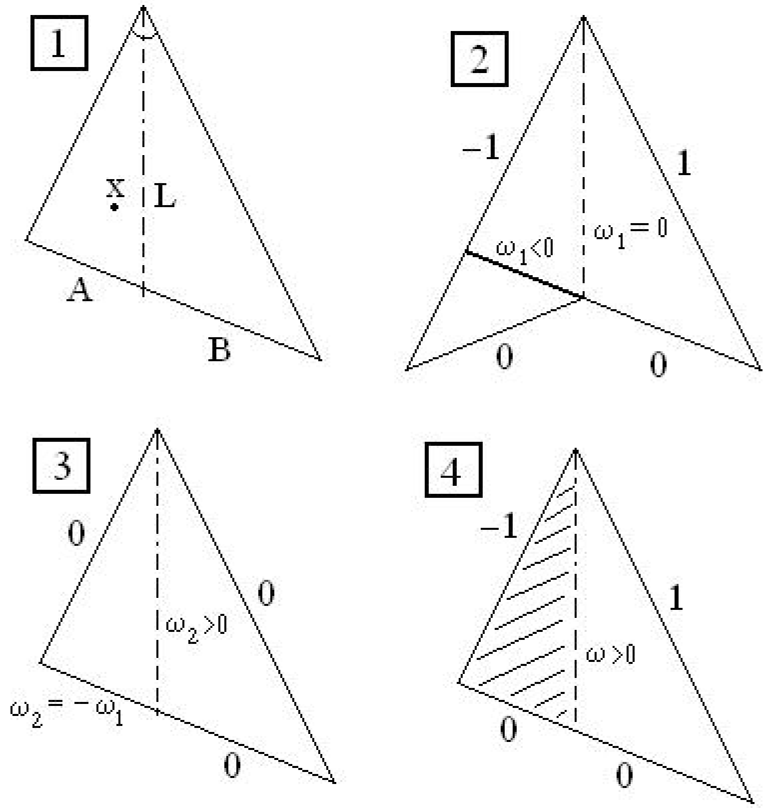

Figure 4.3: Illustration of proof

coordinates express a point inside a triangle as a weighted average of the triangle's vertices. That is, if a point z can be expressed as the weighted average of the three points x_i,

$$z = ax_1 + bx_2 + cx_3,$$

with $a, b, c \in \mathbb{R}$ and $a + b + c = 1$, then we say that z has barycentric coordinates $(a : b : c)$ with respect to the triangle $x_1x_2x_3$. Notice that z is in the interior of the triangle if and only if $a, b, c > 0$.

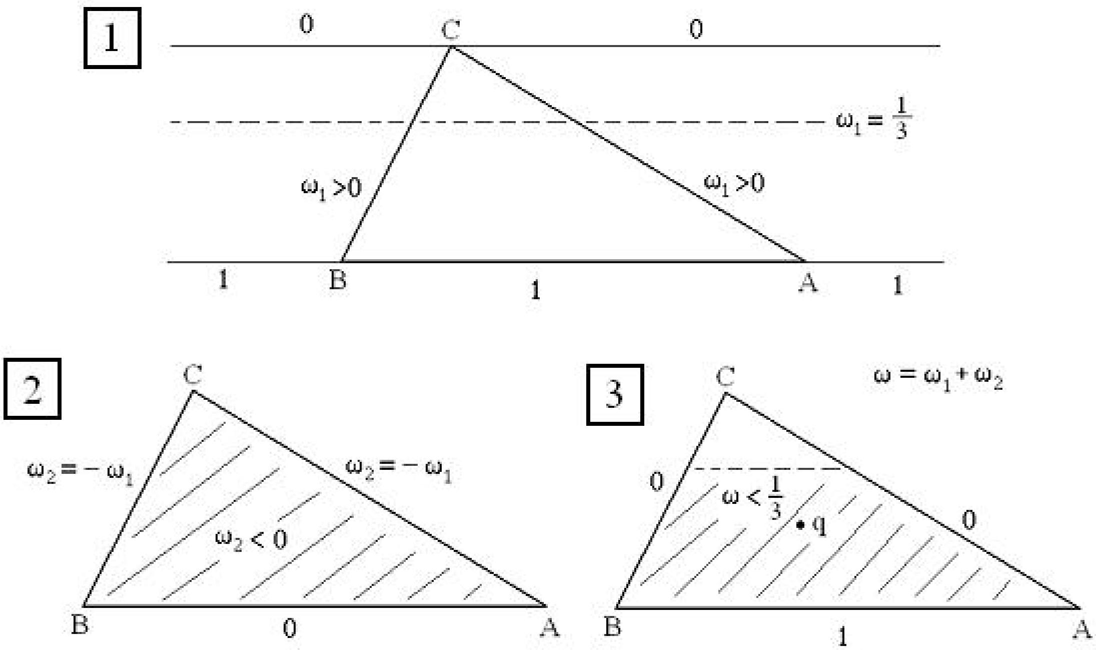

Figure 4.4: Illustration of Proof.

Theorem. *The barycentric coordinates of the conformal center are all less than* $2/3$*.*

Proof. Let a triangle with points A, B, C be given, oriented in the complex plane with side AB along on the real axis. If we normalize the triangle so that vertex C has imaginary part 1, then the corresponding barycentric coordinate of any point C is the imaginary part of that point. Define a harmonic function ω_1 on the infinite strip with boundary values 0 above and 1 below, as shown in Fig. 4.4-1. Solving the Dirichlet problem gives us $\omega_1(z) = \text{Im}(z)$. Therefore, $\omega_1 = 1/3$ on the dashed line with imaginary part $2/3$, and has smaller values above and greater values below. Define another harmonic function ω_2, with $\omega_2 = 0$ on side AB and $\omega_2 = -\omega_1$ on the other sides of the triangle, as in Fig. 4.4-2. Again by the Maximum Principle, $\omega_2 < 0$ along the dashed line. Finally, set

$$\omega = \omega_1 + \omega_2,$$

so that ω is harmonic, and has the boundary values shown in Fig. 4.4-3. Notice that $\omega < 1/3$ along the dashed line, so $\omega = 1/3$ only at points below this line. Since the conformal center q has $\omega(q) = 1/3$, it must lie below the dashed line. Thus, its barycentric coordinate corresponding to vertex C is less than $2/3$. We can similarly show the other two barycentric coordinates are also less than $2/3$. $\square$

Chapter 5

The Schwarz-Christoffel Transformation

These immediate results are nice, but we can be more precise by making use of a powerful complex analysis tool, the Schwarz-Christoffel Transformation. For any given polygon, we can create a Schwarz-Christoffel Transformation from the upper half of the complex plane to the polygon, as in the example shown in the figure:

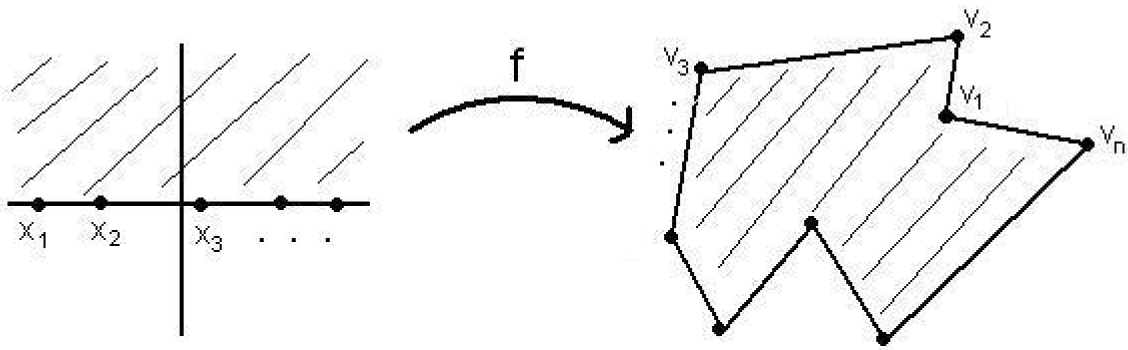

Figure 5.1: Schwarz-Christoffel Transformation f maps the upper half-plane to an arbitrary polygon.

We create this map in the following manner, as described in the appendix of Saff and Snider [8]:

1. Enumerate n vertices of the polygon as v_i in counterclockwise order.

2. Enumerate the corresponding interior angles πa_i.

3. Select $n-1$ distinct points on the real line x_i, so that $x_i < x_{i+1}$.

4. Define the Schwarz-Christoffel Transformation to be

$$f(z) = \int_0^z \prod_{i=1}^{n-1} (\zeta - x_i)^{a_i - 1} d\zeta.$$

This function maps each x_i to the corresponding vertex of the polygon v_i with interior angle a_i. The point at infinity maps to the last vertex, v_n. Only certain selections of points x_i will suffice, but there always exists a possible selection [8].

The Schwarz-Christoffel Transformation is also *conformal* [8]. Recall that a complex map is conformal if it is analytic with non-zero derivative throughout its domain. We also know that harmonic functions are invariant under conformal mapping. Formally

Theorem. *If $\omega : T \to \mathbb{R}$ is a harmonic function, and $g : S \to T$ is a conformal map, then $\omega \circ g : S \to \mathbb{R}$ is also a harmonic function* [1].

By this theorem, the Schwarz-Christoffel Transformation f, and its inverse f^{-1}, preserve harmonic measure. Thus, for any side S of a polygon Ω, we have

$$\omega(z, S, \Omega) = \omega(f^{-1}(z), f^{-1}(S), H_+),$$

at all points $z \in \Omega$ (Here H_+ indicates the upper half of the complex plane). Consequently, if wish to find the the harmonic measure in a polygon, it suffices to find the values in the half-plane, then apply the appropriate Schwarz-Christoffel Transformation.

To demonstrate this technique, let us suppose a triangle is given. After selecting one side S, we wish to find the points in the triangle where the harmonic measure of S is $1/3$. For clarity, denote the values of the harmonic measure $\omega(z, S, \Omega)$ by the shorter notation $\omega(z)$. Note that $\omega(z)$ is a harmonic function of z. We now map the triangle to the half-plane using an inverse Schwarz-Christoffel Transformation, as in Fig. 5.2.

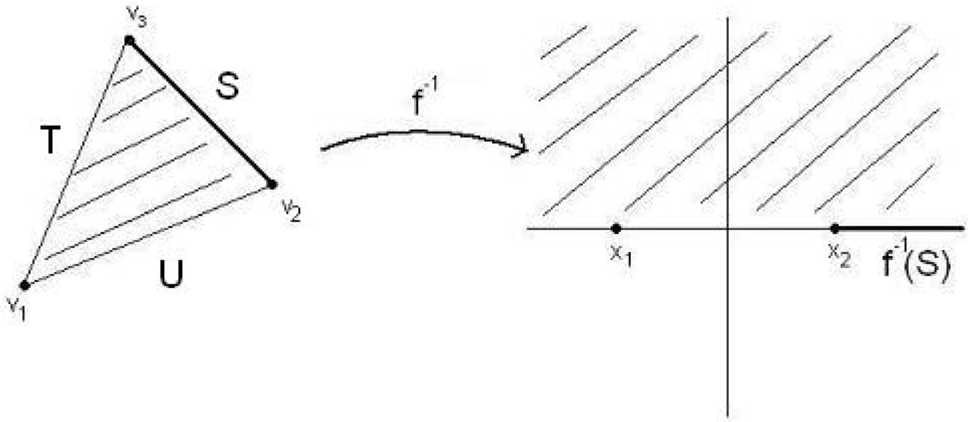

Figure 5.2: The inverse Schwarz-Christoffel Transformation f^{-1} maps a triangle to the half-plane.

We similarly define the function $\omega^*(x)$ according to the harmonic measure in the half-plane, so that

$$\omega^*(x) = \omega\left(x, f^{-1}(S), H_+\right).$$

Because f preserves ω, we have

$$\omega^*\left(f^{-1}(z)\right) = \omega(z)$$

for all points z inside the triangle. Now $\omega^*(x)$ is harmonic, so finding its values at all points in the half-plane is a simple Dirichlet problem. We find that $\omega^*(x) = t$ on the points $x_2 - re^{i\pi t}$ for all $r > 0$ [8]. Thus, $\omega^*(x) = 1/3$ on the line L in Fig. 5.3.

If we repeat this analysis for side T, we find $\omega\left(x, f^{-1}(T), H_+\right) = 1/3$ for points x on the line M. Thus, at the point q, both these harmonic measures equal $1/3$. Since a Brownian path from any point $x \in H_+$ must eventually cross the real axis with probability 1,

$$\omega\left(x, f^{-1}(S), H_+\right) + \omega\left(x, f^{-1}(T), H_+\right) + \omega\left(x, f^{-1}(U), H_+\right) = 1$$

Consequently, we have

$$\omega\left(q, f^{-1}(U), H_+\right) = 1 - 1/3 - 1/3 = 1/3.$$

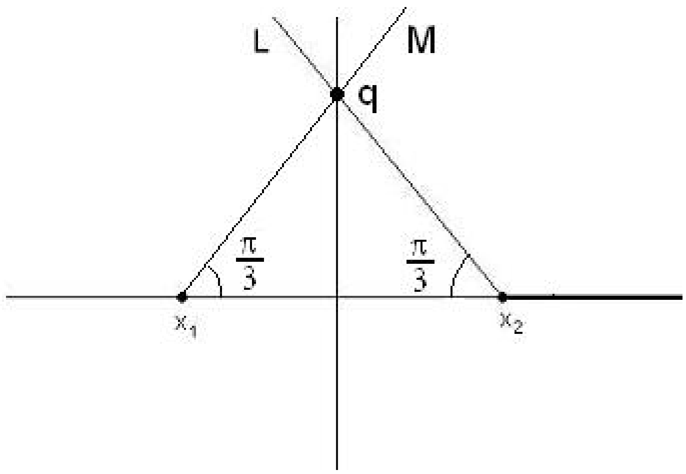

Figure 5.3: The $\omega^* = 1/3$ on the line L.

Thus, all three harmonic measures equal $1/3$ at q in the half-plane. Since f preserves harmonic functions, the harmonic measures of sides S, T, and U all equal $1/3$ at $f(q)$, so $f(q)$ is the conformal center of the triangle Ω.

Theorem. *Every triangle has a conformal center.*

Proof. Let a triangle $v_1v_2v_3$ be given, as in Fig. 5.4. For $x_1 = -1$, $x_2 = 1$, there exists a Schwarz-Christoffel Transformation f with $f(x_1) = v_1$, $f(x_2) = v_2$, and $f(\infty) = v_3$. According to the previous analysis, for $q = \sqrt{3}i$,

$$\omega\left(q, f^{-1}(S), H_+\right) = \omega\left(q, f^{-1}(T), H_+\right) = \omega\left(q, f^{-1}(U), H_+\right) = 1/3.$$

Because f is conformal, it preserves harmonic measure, so that $f(q) = f(\sqrt{3}i)$ is the conformal center of $v_1v_2v_3$. □

Specifically, for a triangle with angles πa, πb, πc, we can choose $x_1 = -1$, $x_2 = 1$, and use the following Schwarz-Christoffel Transformation:

$$f(z) = \int_0^z (\zeta + 1)^{a-1}(\zeta - 1)^{b-1} d\zeta$$

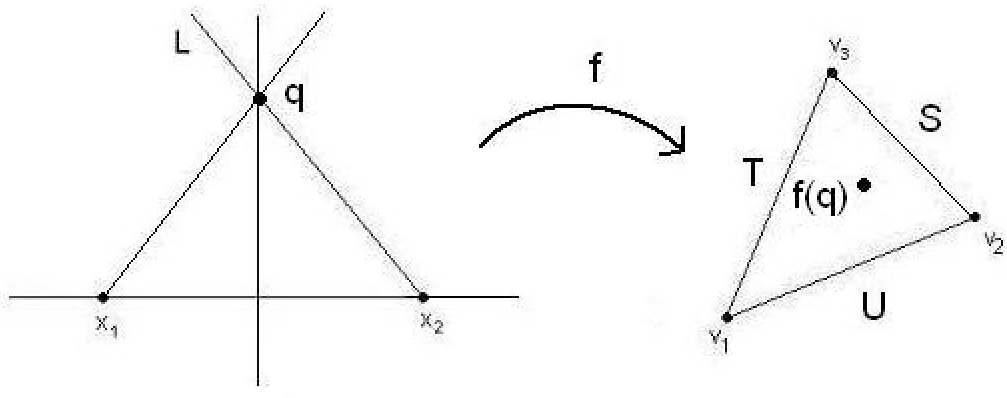

Figure 5.4: We can map the conformal center from the half-plane to locate it within the triangle.

As shown in the proof, the conformal center of the triangle is located at

$$f(\sqrt{3}i) = \int_0^{\sqrt{3}i} (\zeta + 1)^{a-1}(\zeta - 1)^{b-1} d\zeta.$$

As we will show in a later chapter, similar analysis can be performed for polygons other than triangles. However, for an arbitrary n-sided polygon, there is no analogous guarantee to our claim that the lines L and M intersect, as in Fig. 5.3. For a quadrilateral, there are three such lines. Only for certain choices of x_i will these lines all intersect at the conformal center. Our procedure for creating a Schwarz-Christoffel Transformation does not give us complete freedom to choose the points x_i, so there *may not* be a transformation that maps the desired points to the desired vertices. Consequently, a polygon with more than three sides does not generally have a conformal center.

Chapter 6

Geometric Triangle Centers

We have now shown that every triangle has a conformal center. This much was known before we began our work. However, before our research, the conformal center had only appeared as an incidental point in the discussion of conformal maps, never as the focus of attention as a triangle center. We would like to know how it relates to known triangle centers. A vast amount of work has been done in this field. A website called the "Encyclopedia of Triangle Centers," maintained by Clark Kimberling at the University of Evansville, lists 1114 known geometric triangle centers [7]. One natural question is, "Is the conformal center merely another of these known triangle center in disguise?"

Without knowing the precise location of the conformal center, this question is difficult to answer. Fortunately, we *can* exactly determine the location of the conformal center in two special limiting cases. By comparing its behavior to that of other triangle centers, we can distinguish the conformal center from many known triangle centers.

The first extremal case we consider is a very flat isosceles triangle, shown in Fig. 6.1. As we "squish" an isosceles triangle, making it infinitely flat and long, the barycentric coordinates of the conformal center (corresponding to vertices z_1, z_2, z_3) approach

$$C = \left(\frac{2}{3} : \frac{1}{6} : \frac{1}{6}\right).$$

The barycenter, on the other hand, remains at

$$B = \left(\frac{1}{3} : \frac{1}{3} : \frac{1}{3}\right),$$

as shown in Fig. 6.1. We therefore conclude that the conformal center is distinct from the barycenter, since they diverge in this limiting case.

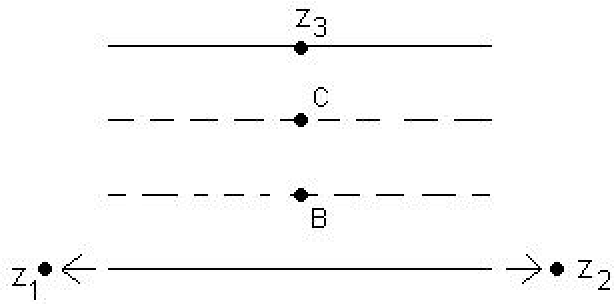

Figure 6.1: In a flat isosceles triangle, the barycenter B converges to a different point than the conformal center C.

The other limiting case is an infinitely tall isosceles triangle. As the height of the triangle increases, its interior approaches a semi-infinite strip, and the conformal center converges to a point. To find this point, note that the inverse sine function maps the half-plane to the semi-infinite strip, as shown: Since the sine function is

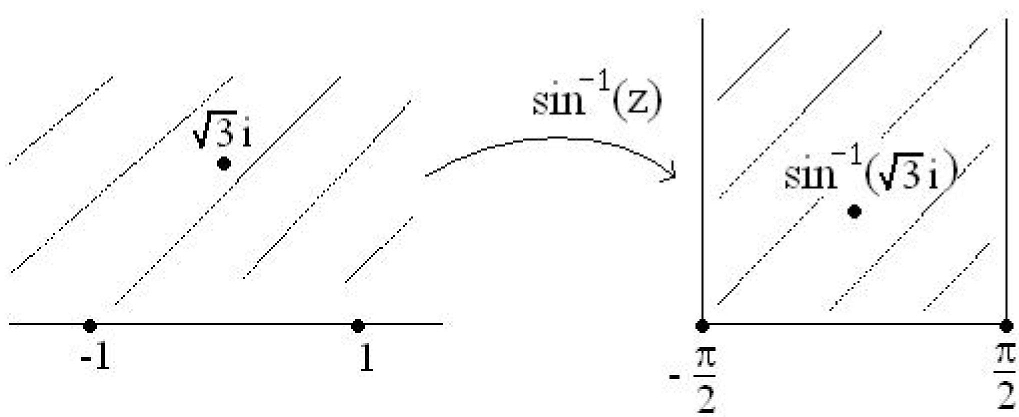

Figure 6.2: The inverse sine function maps the half-plane to the semi-infinite strip.

conformal, it preserves the conformal center. Therefore, the conformal center of the this strip is located at

$$\sin^{-1}\left(\sqrt{3}i\right) = i\sinh^{-1}\left(\sqrt{3}\right) = i\ln\left(2+\sqrt{3}\right)$$

This extremal case can also be used to rule out various geometric centers. For example, the barycenter diverges to infinity in the half-strip, so it clearly can't be the conformal center. The incenter occurs at the intersection of the angle bisectors, so it lies at $\pi i/2$, and it also cannot be the conformal center. This technique is effective, but checking each triangle center individually is very time consuming.

Fortunately, we can use *triangle functions* along with a tool provided by the Encyclopedia of Triangle Centers to check all 1114 known centers at once [7]. First we need a few definitions.

Definition. *A triangle with angles* $(\pi a, \pi b, \pi c)$ *is said to have* ***normalized angles*** (a, b, c)*. Notice that* $a + b + c = 1$ [7].

Definition. *If a point* p *and triangle* ABC *are given, and* p *lies a distance* x *from the side* BC*,* y *from the side* CA*, and* z *from the side* AB*, then the* ***trilinear*** *coordinates of* p *can be written as* $(kx : ky : kz)$ *for any non-zero constant* k [7].

Definition. *A point* z *is a* ***triangle center*** *if there exists some* ***triangle function*** $f(x, y, z)$ *defined for* $x, y, z \in [0, 1]$ *with* $x+y+z = 1$*, such that in a triangle with normalized angles* (a, b, c)*, the barycentric coordinates of* z *are* $(f(a, b, c) : f(b, c, a) : f(c, a, b))$ [7].

Actually, Kimberling's definition requires that f be a function of the side lengths. Our definition is equivalent, and better for our purposes. To illustrate this definition, notice that the barycenter is a triangle center with triangle function $f(a, b, c) = 1/3$ for all values of a, b, c. The incenter has $f(a, b, c) = k \csc a$, where k is a normalizing constant. Notice that each coordinate of the incenter is proportional to the length of the opposite side. In Chapter 7 we determine the triangle function for the conformal center.

Chapter 7

Triangle Function of the Conformal Center

In this chapter we show how to convert the integral expression for the conformal center derived in Chapter 5 into a triangle function $f(a,b,c)$. This analysis is taken from our forthcoming paper, *The Conformal Center of a Triangle* [5], so we give only an outline of the steps here.

We begin with the integral expression for the conformal center, from Chapter 5:

$$f(\sqrt{3}i) = \int_0^{\sqrt{3}i} (\zeta+1)^{a-1}(\zeta-1)^{b-1} d\zeta$$

By changing variables by $\zeta = \tan\theta$, we obtain:

$$f(\sqrt{3}i) = ie^{\pi i(b-1)} \int_0^{\pi/3} e^{i(a-b)\theta} \cos^{c-1}\theta d\theta$$

Now barycentric coordinates can be expressed as a ratio of areas. That is, the coordinate of the point P corresponding to vertex A in Fig. 7.1 is the ratio $\frac{Area(PBC)}{Area(ABC)}$ [7]. Using this technique, and the previous equation, we obtain the following triangle function $f(a,b,c)$ for the conformal center. Recall that this function gives the barycentric coordinates of the conformal center in terms of the normalized angles a, b, c.

$$f(a,b,c) = \frac{2^a\Gamma(1-b)\Gamma(1-c)}{\pi\Gamma(a)} \int_0^{\pi/3} \cos\left((b-c)\theta\right) \cos^{a-1}\theta d\theta$$

By expanding the cosines into exponentials and applying the binomial theorem, we obtain a convergent series in the integrand. We evaluate the integral for each term, which turns our integral expression into the following summation.

$$f(a,b,c) = \frac{\Gamma(1-b)\Gamma(1-c)}{\pi\Gamma(a)} \sum_{m=0}^{\infty} \binom{a-1}{m} \left(\frac{\sin\left(\frac{2\pi}{3}(m+b)\right)}{m+b} \right)$$

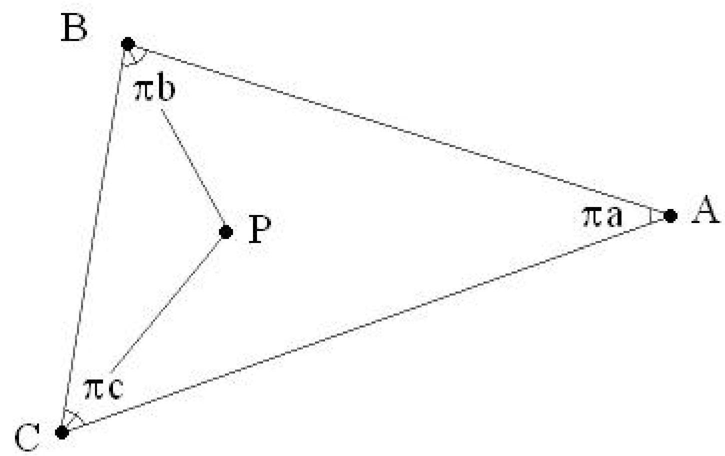

Figure 7.1: The barycentric coordinate of P for vertex A is $\frac{Area(PBC)}{Area(ABC)}$.

After further rearrangement, we discover the following recurrence relation:

$$f(a,b,c) = \frac{B(1-b,1-c)}{\pi} \sin\left(\frac{\pi}{3}(b-c-1)\right) + f(a+1,b-1,c)$$

Using the fact that $f(a,b,c) = f(a,c,b)$, we can obtain a similar result. By combining the two expressions, we get a recurrence for $f(a,b,c)$ in terms of $f(a+2,b-1,c-1)$. It can be shown that $f(a,b,c)$ converges to 1 as $a \to \infty$ and $b,c \to -\infty$. Therefore, by repeatedly applying this recurrence and telescoping, we obtain

$$\begin{aligned} f(a,b,c) &= 1 + \left(\sum_{k=0}^{\infty} \frac{B(1-b+k,1-c+k)}{1+a+2k}\right)\left(\frac{b-c}{2\pi}\right)\sin\left(\frac{\pi}{3}(b-c)\right) \\ &\quad - \left(\sum_{k=0}^{\infty} B(1-b+k,1-c+k)\right)\left(\frac{\sqrt{3}}{2\pi}\right)\cos\left(\frac{\pi}{3}(b-c)\right). \end{aligned}$$

We can clean this up a bit by defining

$$\Phi(x,y) = \sum_{k=0}^{\infty} B(x+k,y+k)$$

$$\Psi(x,y) = \sum_{k=0}^{\infty} \frac{B(x+k,y+k)}{x+y+2k},$$

If we insert these definitions, use the fact that $a+b+c=1$, and apply some trigonometric identities, we obtain

$$f(a,b,c) = \frac{\sin(\pi a)}{\pi \sin(\pi b)} \left[\sin\left(\frac{\pi}{3}\left(b-c+\frac{1}{2}\right)\right)\left(\frac{\sqrt{3}}{2}\right)\Phi(1-a,1-c) \right.$$
$$\left. + \cos\left(\frac{\pi}{3}\left(b-c+\frac{1}{2}\right)\right)\left(\frac{c-a}{2}\right)\Psi(1-a,1-c)\right].$$

This last expression converges like $1/4^k$ after k terms. This allows the Java program described in Chapter 8 to operate much more quickly than if it had to numerically evaluate the original integral expression, or sum the earlier series expressions.

Chapter 8

Java Program

To experiment with the conformal center, we built a Java program that allows a user to visually manipulate the vertices of a triangle, then plot various triangle centers for that triangle. The interested reader can experiment with this program online at www.math.hmc.edu/∼aiannaccone. A screen shot is shown in Fig. 8.1.

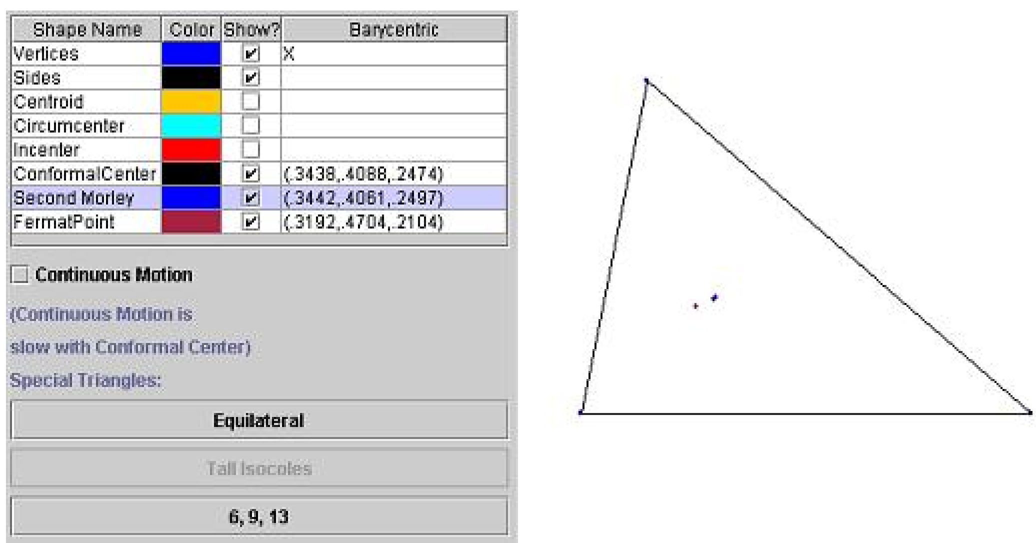

Figure 8.1: The user can move the triangle's vertices, and the Java program displays the selected triangle centers along with their barycentric coordinates.

Here the user has displayed the conformal center, Fermat point, and Second Morley point. The user can observe how close the points are, and how their distance changes for different triangles.

Using this program along with the Encyclopedia of Triangle Centers [7], we were able to compare the conformal center to many other geometric triangle centers. The Encyclopedia of Triangle Centers has a list of the first trilinear coordinate of each center for a triangle with side lengths $6, 9, 13$. That is, it has an ordered list of the first trilinear for the incenter, barycenter, etc. This means that if we can calculate the first trilinear for a point of interest in the $6, 9, 13$ triangle, we can see if it coincides with any known triangle center, thus checking against all 1114 listed centers at once. Using the formula from Chapter 7, we calculated the first trilinear coordinate of the conformal center for the $6, 9, 13$ triangle. By then checking our value with those in the Encyclopedia of Triangle Centers list, we discovered that the conformal center does not coincide with any of those 1114 triangle centers.

While experimenting with the Java program, we observed that the conformal center generally lies fairly close to the incenter. Now the incenter is the center with barycentric coordinates proportional to the lengths of the opposite sides, which we call the "side-weighted center." Another known center is the "angle-weighted center" (also known as the Hofstadter Zero point [7]), which has barycenter coordinates proportional to the corresponding angles. We observed that this point and the incenter tend to flank the conformal center, with the conformal center roughly two-thirds of the way from angle-weighted center to the incenter. We calculated the first trilinear coordinate for this third point, and used the Encyclopedia of Triangle Centers to determine it was the Second Morley point. Further experimentation revealed that the Second Morley point always lies close to the conformal center.

Chapter 9

Closeness to the Second Morley Point

The Second Morley point is an interesting triangle center. Its construction, which is relatively simple among the less well-known triangle centers, is shown in Fig. 9.1. The trisectors of the three angles meet at A', B', and C'. These three points make an equilateral triangle, whose center is the first Morley point. The lines AA', BB', CC' all coincide at a point, called the Second Morley point.

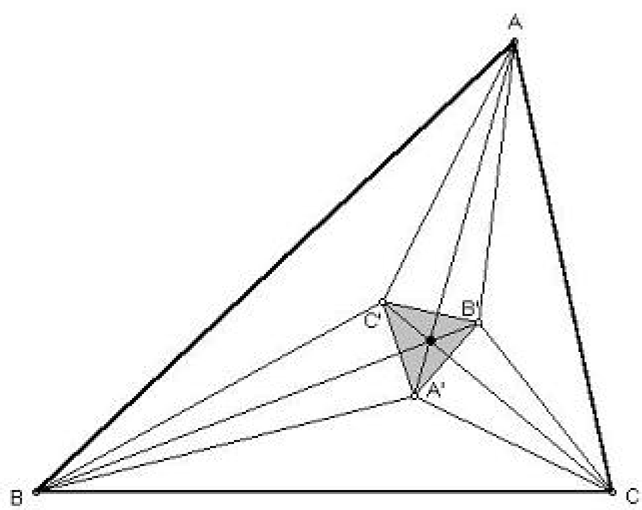

Figure 9.1: Construction of the Second Morley point (from the Encyclopedia of Triangle Centers [7]).

The Java program suggested to us that the Second Morley point and the conformal center were often very close. Using Maple, we calculated these points for a wide range of triangles. We found that for every triangle we considered, the points were strikingly close together.

To better quantify the distance between the two points, we introduced the following metric for the distance between two points p and q in a triangle:

$$d(p,q) = \sqrt{\frac{\delta(p,q)^2}{A}}$$

where $\delta(p,q)$ is the Euclidean distance between the points, and A is the area of the triangle. Note that this distance is area-normalized, and therefore does not change when we dilate, translate, or rotate the triangle. Over the range of triangles we computed, the area-normalized distance between the two points was never greater than $.016$.

Another expression of this closeness comes from drawing lines from a vertex to the two points, and determining the angle between them. To make our expressions simpler, we'll use the Fermat point as a reference. Recall that the Fermat point of a triangle is the point from which line segments to each vertex meet at angles of $2\pi/3$. Let us draw a line from vertex C to the Fermat point, then draw a line from C to the conformal center, as in Fig. 9.2. Call the difference in these angles η_c. Similarly, we may draw a line from C to the Second Morley point, and define the angle difference between that point at the Euler point as λ_c.

In *The Conformal Center of a Triangle* [5], we show that:

$$\tan \eta_c = \frac{(b-a)\Psi(1-a,1-b)}{\sqrt{3}\Phi(1-a,1-b)}$$

$$\tan \lambda_c = \frac{\sin\left(\frac{\pi}{3}(b-a)\right)}{\cos\left(\frac{\pi}{3}(b-a)\right) + 2\cos\left(\frac{\pi}{3}(1-c)\right)}$$

Notice that bound on the difference $\eta_c - \lambda_c$ defines a wedge-shaped region in which both points must be located, and vice-versa. Since the points lie close together, we expect this wedge-shaped region to be a small fraction of the angle πc. Indeed, we observe

$$|\eta_c - \lambda_c| < .0157(\pi c)$$

over the same range of triangles considered above.

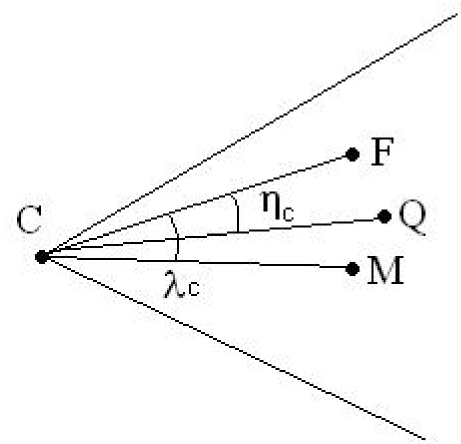

Figure 9.2: Here η_c and λ_c measure the angles between the Fermat point F and the conformal center Q and Second Morley point M, respectively. For clarity, this is *not* drawn to scale. For a more accurate representation, see Fig. 8.1

Chapter 10

Polygons with Conformal Centers

As we showed in Chapter 5, all triangles have a conformal center, but other polygons generally do not. Which special polygons have a conformal center? In this chapter, we use the Schwarz-Christoffel Transformation to answer this question. We begin by establishing some terminology.

Definition. *We call an ordered set* $S = (x_1, x_2, \ldots, x_m)$ *an* ***m-point partition*** *of the upper half-plane if* $x_i \in \mathbb{R}$ *and* $x_i < x_{i+1}$.

Definition. *We say* q *is the* ***conformal center*** *of* H_+ *with the partition* $S = (x_i)$ *if a Brownian path starting at* q *is equally likely to first exit* H_+ *through any of the segments*

$$(-\infty, x_1), (x_1, x_2), \ldots, (x_n, \infty).$$

Lemma. *For* $m \geq 2$*, there is exactly one* m*-point partition that has a conformal center, subject to translation and dilation.*

Proof. For existence, let $q = i$, and for $k = 1 \ldots n$, let

$$x_k = -\cot\left(\frac{\pi k}{m+1}\right).$$

Now for $k = 1, \ldots, m$, define the interval $I_k = (-\infty, x_k)$. Because of our choice of x_k, the line from x_k to the conformal center q makes an angle of $2\pi/(m+1)$ with the real line, as seen in Fig. 10.1. It follows, then, that

$$\omega(q, I_k, H_+) = \frac{1}{m+1}.$$

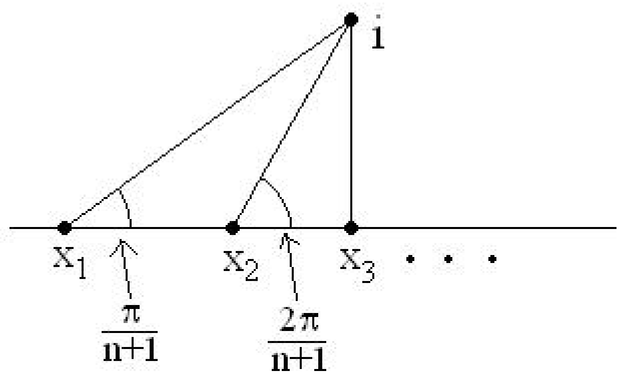

Figure 10.1: The line from x_k to the conformal center at i makes an angle $\frac{k\pi}{m+1}$.

That is, the probability that a Brownian path starting at q first exits the half-plane through the interval $I_1 = (-\infty, x_1)$ is $1/(m+1)$. The probability that it exits through (x_1, x_2) is therefore

$$\begin{aligned} \omega(q, I_2 \setminus I_1, H_+) &= \omega(q, I_2, H_+) - \omega(q, I_1, H_+) \\ &= \frac{2}{m+1} - \frac{1}{m+1} \\ &= \frac{1}{m+1}. \end{aligned}$$

For all k, the reasoning is similar and the result is the same. Thus, by the definition above, q is the conformal center of the H_+ with the partition $S = (x_k)$.

To show uniqueness, suppose two partitions $S = (x_i)$ and $S' = (x_i')$ are given, and that H_+ has conformal centers q and q' with these partitions, respectively. Let us scale and translate S' so that $q = q'$. Assume for contradiction that $x_1 \neq x_1'$. Then the harmonic measures for the intervals $I_1 = (-\infty, x_1)$ and $I_1' = (-\infty, x_1')$ are not equal. That is, we have

$$\omega(q, I_1, H_+) \neq \omega(q', I_1', H_+)$$

It follows that one of these points cannot be a conformal center, which is a contradiction. Our assumption must therefore be false, so $x_1 = x_1{}'$. By the same reasoning, we can successively show $x_i = x_i'$ for $i = 1, \dots, m$. This means that $S = S'$, subject to scaling and translation, as desired. □

Suppose we create a Schwarz-Christoffel Transformation f that maps the points x_i to the vertices v_i of a polygon. Because f is conformal, it preserves harmonic measure. Therefore, the polygon has a conformal center if and only if the half-plane has a conformal center with partition (x_i). This fact yields the following theorem.

Theorem. *Let $S = (a_i)$ be an ordered set of $n \geq 3$ angles such that $\sum a_i = (n-2)\pi$. Then there is exactly one polygon, with has angles a_i in clockwise order, that has a conformal center.*

Proof. First we show existence of such a polygon. By the previous theorem, there is exactly one $(n-1)$-partition for which the half-plane has a conformal center. Recall that the construction of the Schwarz-Christoffel Transformation from Chapter 5 allows us to choose the interior angles of the polygon. Therefore, by applying a Schwarz-Christoffel Transformation to this partition, we obtain one polygon with the desired angle set. (Notice, however, that we have no control over the side lengths in this polygon).

To show uniqueness, suppose there are two such polygons. Again by the construction of the Schwarz-Christoffel Transformation in Chapter 5, we can find two partitions which the Schwarz-Christoffel Transformations will map to these polygons. Both of these partitions have a conformal center, so by the above lemma, they are identical, subject to translation and dilation. The polygons produced by applying the Schwarz-Christoffel Transformation to these partitions are therefore identical. That is, if two polygons each have a conformal center, then they are identical, subject to translation and dilation. □

This result implies that by considering every angle set, we can use the Schwarz-Christoffel Transformation to numerically obtain all the polygons with conformal centers. However, given any set of angles, the determination of the polygon's side lengths must be made using the unpleasant integrals generated by the Schwarz-Christoffel Transformation. Thus, although we know the angles of a polygon with a conformal center, we may not be able to determine a closed-form solution for its side lengths and coordinates.

Chapter 11

The Conformal Center of a Quadrilateral

For every set of angles that can form a closed polygon, there is a polygon with those angles and a conformal center. It would be nice, however, if we could locate the conformal center within such a polygon. Using the Schwarz-Christoffel Transformation, we can find an integral expression for its coordinates, as we did with the triangles. In the case of triangles, this expression was manageable enough to turn into a series expression, and from it we were able to deduce some interesting series. The general case for a polygon is much more challenging, but if we restrict ourselves to quadrilaterals, the formula is somewhat manageable. For certain special quadrilaterals, we can express the location of the conformal center with a convergent series.

First consider the general case, where our quadrilateral has an angle set (a, b, c, d) and there is no known relationship between these angles, other than $a+b+c+d = 2$. We may generate the desired polygon by using the Schwarz-Christoffel Transformation on the half-plane with the partition $S = (-1, 0, 1)$. With this partition, the half-plane has conformal center at i. This appropriate transformation is

$$f(z) = \int_0^z (\zeta + 1)^{a-1} \zeta^{b-1} (\zeta - 1)^{c-1} d\zeta.$$

This transformation maps the partition points to the four vertices:

$$f(-1), f(0), f(1), f(\infty),$$

and the conformal center maps to

$$f(i) = \int_0^i (\zeta + 1)^{a-1} \zeta^{b-1} (\zeta - 1)^{c-1} d\zeta.$$

This expression can be simplified by the substitution $u = -i\zeta$, which yields

$$\begin{aligned} f(i) &= \int_0^1 (iu+1)^{a-1}(iu)^{b-1}(iu-1)^{c-1} \\ &= \int_0^1 (u^2+1)^{\frac{a-1}{2}+\frac{c-1}{2}} \left(\frac{iu+1}{\sqrt{u^2+1}}\right)^{a-1} \left(\frac{iu-1}{\sqrt{u^2+1}}\right)^{c-1} i^b u^{b-1} du \end{aligned}$$

We can simplify this using

$$\frac{iu+1}{\sqrt{u^2+1}} = e^{i\tan^{-1}(u)}$$

$$\frac{iu-1}{\sqrt{u^2+1}} = e^{i(\pi-\tan^{-1}(u))}$$

Next, we substitute $\theta = \tan^{-1} u$ to obtain

$$f(i) = e^{i\pi(c+b/2-1)} \int_0^{\pi/4} \cos^{d-1}\theta \sin^{b-1}\theta e^{i(a-c)\theta} d\theta$$

Notice that the vertex in the quadrilateral opposite the point at 0 is mapped from the point at infinity in the half-plane. We can therefore find it, using a similar analysis, to be

$$f(i\infty) = e^{i\pi(c+b/2-1)} \int_0^{\pi/2} \cos^{d-1}\theta \sin^{b-1}\theta e^{i(a-c)\theta} d\theta.$$

11.1 Symmetric Quadrilaterals

As shown in Chapter 10, we know that for each angle set, there is exactly one polygon with a conformal center. However, there are infinitely many polygons with such an angle set, so we can't yet identify the quadrilaterals that have a conformal center. For instance, both the quadrilaterals shown in Fig. 11.1 have angle set (q, p, q, r). We can't yet determine which, if either, has a conformal center. The following theorem, however, shows that every symmetric quadrilateral has a conformal center. These symmetric quadrilaterals look like kites and Starfleet-symbols, as shown in Fig. 11.2.

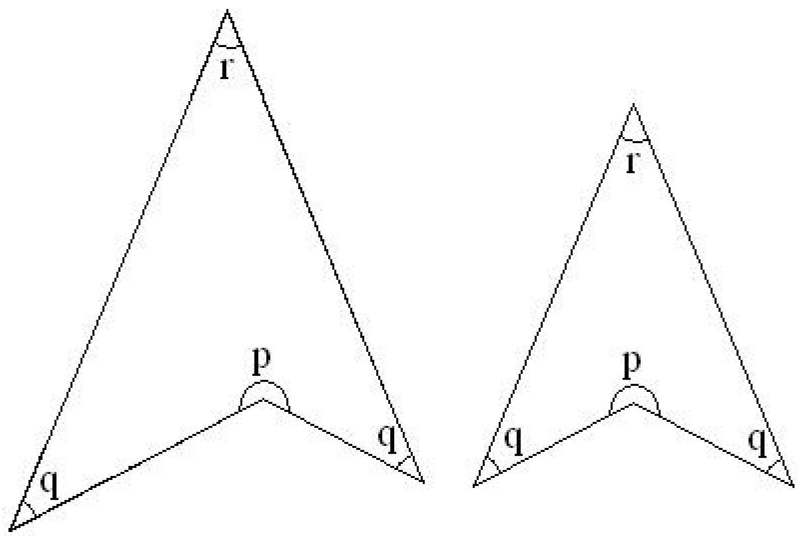

Figure 11.1: Distinct quadrilaterals with the same angle set (q, p, q, r).

Theorem. *If a quadrilateral has two of its vertices on a line, and is symmetric about that line, then it has a conformal center.*

Proof. Let x, y be the vertices on the axis of symmetry. Denote the sides adjacent to x as A_1 and A_2, and the sides adjacent to y as B_1 and B_2, as in Fig. 11.2. By symmetry of the harmonic measure, for every point z on the axis of symmetry (dotted line), we have

$$\omega(z, A_1, \Omega) = \omega(z, A_2, \Omega)$$

$$\omega(z, B_1, \Omega) = \omega(z, B_2, \Omega)$$

Let us parameterize this axis as $z(t) = x + t(y - 1)$, so that $z(0) = x$ and $z(1) = y$. Also let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. For $t \in [0, 1]$, define a function

$$f(t) = \omega\left(z(t), B, \Omega\right) - \omega\left(z(t), A, \Omega\right)$$

Notice, then, that

$$f(0) = \omega\left(x, B, \Omega\right) - \omega\left(x, A, \Omega\right) = -1$$

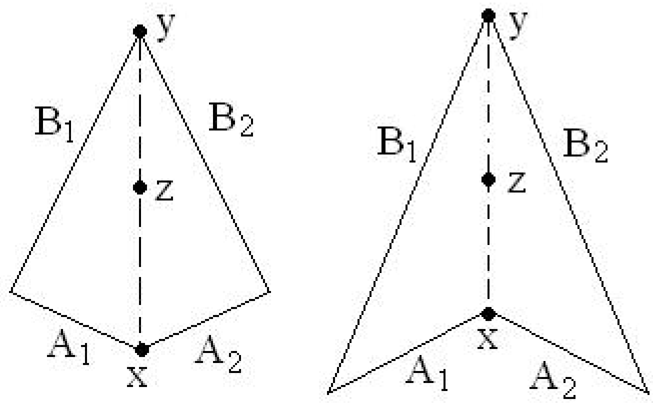

Figure 11.2: Illustration for proof. Symmetric quadrilaterals look like kites and Starfleet-symbols.

$$f(1) = \omega\left(y, B, \Omega\right) - \omega\left(y, A, \Omega\right) = 1.$$

Now $f(t)$ is continuous in t because $z(t)$ is continuous in t, and the harmonic measure functions are continuous in z. We may invoke therefore invoke the Intermediate Value Theorem. This means there is some $c \in (0,1)$ such that $f(c) = 0$. For this value, we have

$$\omega\left(z(c), A, \Omega\right) = \omega\left(z(c), B, \Omega\right),$$

and since $z(c)$ is on the axis of symmetry, we also have

$$\omega\left(z(c), A_1, \Omega\right) = \omega\left(z(c), A_2, \Omega\right)$$

$$\omega\left(z(c), B_1, \Omega\right) = \omega\left(z(c), B_2, \Omega\right)$$

Consequently, the harmonic measure at $z(c)$ is equal for all four sides, and so $z(c)$ is the conformal center. $\square$

Thus, every symmetric quadrilateral has a conformal center. In fact, we can locate this center, because the general quadrilateral integral simplifies in this case. Using the expression from the previous section with

$$a = q,\ b = p,\ c = q,\ d = r,$$

we obtain the following expressions for the conformal center and the vertex at angle d:

$$f(i) = i^{q-1+\frac{p}{2}} \int_0^{\pi/4} \cos^{r-1}\theta \sin^{p-1}\theta d\theta$$

$$f(i\infty) = i^{q-1+\frac{p}{2}} \int_0^{\pi/2} \cos^{r-1}\theta \sin^{p-1}\theta d\theta$$

Note that the constant coefficients are the same. Since the Schwarz-Christoffel Transformation the vertex with angle b to 0, this means that the $f(i)$ and the vertices of angles b and d are collinear. The actually value of the coefficients is unimportant, since it merely indicates a rotation of the polygon. We are truly concerned only with the lengths $|f(i)|$ and $|f(i\infty)|$.

The first of these integrals is just a Beta function. That is,

$$|f(i\infty)| = B\left(\frac{p}{2}, \frac{r}{2}\right)$$

The second integral is more difficult, but can be expressed as a series as follows. Let $G(a,b)$ be defined as

$$G(a,b) = \int_0^{\pi/4} \sin^a u \cos^b u du$$

Then we can differentiate with respect to u in two different ways to obtain the following two expressions:

$$\frac{d}{du}G(a,b) = \sin^a\frac{\pi}{4}\cos^b\frac{\pi}{4} = 2^{-\frac{a+b}{2}},$$

and

$$\begin{aligned} \frac{d}{du}G(a,b) &= \int_0^{\pi/4} a\sin^{a-1} u \cos^{b+1} u du + \int_0^{\pi/4} (-b)\sin^{a+1} u \cos^{b-1} u du \\ &= aG(a-1,b+1) - bG(a+1,b-1). \end{aligned}$$

Combining these expressions yields the recurrence

$$G(a-1,b+1) = \frac{1}{a}\left(2^{-\frac{a+b}{2}} + bG(a+1,b-1)\right),$$

which we rewrite as

$$G(a,b) = \frac{1}{a+1}\left(2^{-\frac{a+b}{2}} + (b-1)G(a+2,b-2)\right).$$

We would like to use this recurrence to obtain a series expression for $G(a,b)$. To do this, we must first show that $\lim_{k\to\infty} G(a+k, b-k)$ converges. We do this as follows:

Notice that, for $u \in (0, \pi/4)$, $\tan u < 1$. In the limit, then,

$$\lim_{k\to\infty} \sin^k u \cos^{-k} u = 0.$$

This limit for the integrand in $G(a,b)$ gives us a limit on $G(a,b)$ itself, since the integration is performed over a compact domain:

$$\lim_{k\to\infty} G(a+k,b-k) = \int_0^{\pi/4} \lim_{k\to\infty}\left(\sin^{a+k} u \cos^{b-k} u\right) du = \int_0^{\pi/4} 0 \cdot du = 0$$

Thus, we may use the recurrence to obtain a series expression, as follows:

$$\begin{aligned}
G(a,b) &= \frac{1}{a+1}\left(2^{-\frac{a+b}{2}} + (b-1)G(a+2,b-2)\right) \\
&= \frac{1}{a+1}\left(2^{-\frac{a+b}{2}} + (b-1)\frac{1}{a+3}\left(2^{-\frac{a+b}{2}} + (b-3)G(a+4,b-4)\right)\right) \\
&= \frac{1}{a+1}2^{-\frac{a+b}{2}} + \frac{b-1}{(a+1)(a+3)}2^{-\frac{a+b}{2}} + \frac{(b-1)(b-3)}{(a+1)(a+3)}2^{-\frac{a+b}{2}}G(a+4,b-4) \\
&\vdots \\
&= 2^{-\frac{a+b}{2}}\left(\frac{1}{a+1}\right)\left(1 + \sum_{k=1}^{\infty} \frac{(b-1)(b-3)\ldots(b+1-2k)}{(a+3)(a+5)\ldots(a+1+2k)}\right)
\end{aligned}$$

Thus, in this special symmetric case, we can find the position of the conformal center within a quadrilateral as a series expression. There may also be other special cases where the elliptic integral simplify, such as trapezoids.

Chapter 12

The Conformal Center of a Circle

Thus far, we have only considered the conformal center in the context of polygons. But for any complex region that is bounded by a Jordan curve, we can divide the boundary into segments and speak of the probability that a Brownian path first exits the domain through that segment. For example, suppose a Brownian path starting at q is equally likely to exit Ω through any of the segments A, B, or C. Then we call q the conformal center of Ω with the 3-point partition (x, y, z). As before with triangles, it can be shown that such a conformal center must exist.

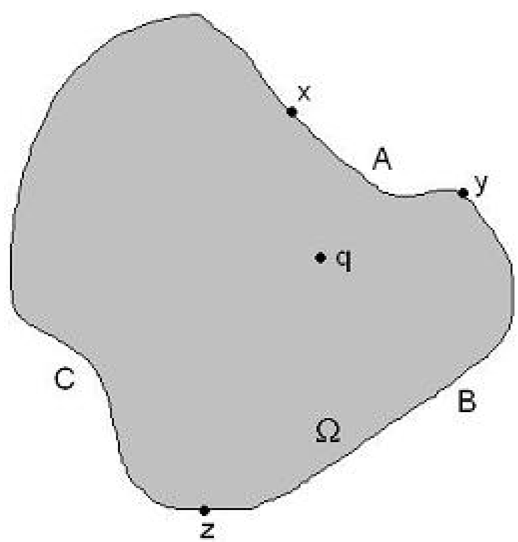

Figure 12.1: A Brownian path from q is equally likely to exit Ω through sides A, B, or C. We call q the conformal center of Ω with the partition (x, y, z).

12.1 Circles with 3-point Partitions

Suppose the domain is a circle, and the partition has three points. The reader may recall that for any two sets of three distinct points, (z_1, z_2, z_3) and (z'_1, z'_2, z'_3), there exists a Möbius transformation that maps each z_i to the corresponding z'_i. Furthermore, it is known that Möbius transformations are conformal, and that they map circles and lines to circles and lines [8]. Consequently, for any three points on the circle, there exists a Möbius transformation to the three roots of unity, $1, \omega, \bar{\omega}$. Since the three original point lie on a circle, as do the three new points, the Möbius transformation sends the entire old circle to the entire new circle. The final map f is as shown:

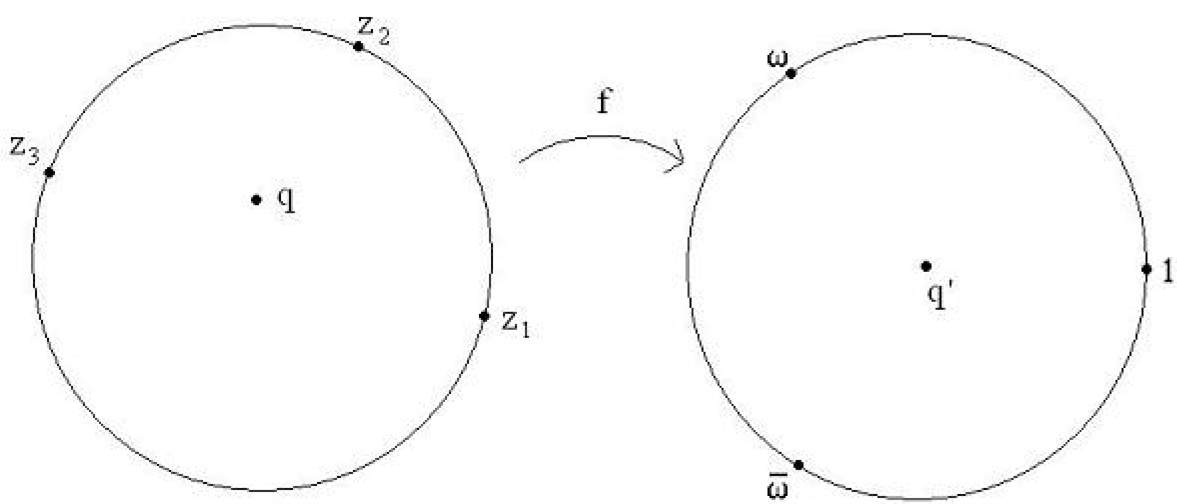

Figure 12.2: A Möbius transformation maps the three points on the left circle to the three points on the right.

Since the 3-point circle with partition $(1, \omega, \bar{\omega})$ has a conformal center at 0, and f is conformal, the original 3-point circle must have a conformal center q. Furthermore, there is a relatively simple way to find the point q. The cross-ratio of four points (a, b, c, d) is defined as follows:

$$a : b : c : d = \frac{(a-b)(c-d)}{(a-d)(b-c)}.$$

Now Möbius transformations preserve cross-ratios [8], so for any conformal map f, we have

$$a : b : c : d = f(a) : f(b) : f(c) : f(d).$$

In our case, this implies

$$q : z_1 : z_2 : z_3 = 0 : 1 : \omega : \bar{\omega}$$

Evaluating both sides of this expression, we obtain

$$\frac{(q-z-1)(z_2-z_3)}{(q-z_3)(z_2-z_1)} = -\bar{\omega}$$

To locate the conformal center q with respect to the triangle $z_1z_2z_3$, we now find its trilinear coordinates. Using the cross-ratio expression, we can determine that the trilinears are proportional to

$$\left(\sin\left(\pi a + \frac{\pi}{3}\right) : \sin\left(\pi b + \frac{\pi}{3}\right) : \sin\left(\pi c + \frac{\pi}{3}\right)\right)$$

By computing the first trilinear coordinate for this point in the $6, 9, 13$ triangle, we were able to compare it to the collection of known triangle centers in the Encyclopedia of Triangle Centers. We found that this first trilinear coordinate equals the corresponding coordinate for the First Isodynamic point. We looked up the triangle function for the First Isodynamic point and found it matched the trilinears above. That is, the conformal center of a circle with a 3-point partition coincides with the First Isodynamic point of the triangle whose vertices are those three points.

12.2 Circles with 4-point Partitions

In the case of a 4-point circle, we find the circle does not always have a conformal center. The analysis used in the 3-point case does not hold because we cannot necessarily find a Möbius transformation that maps for four partition points to $(1, i, -1, -i)$, as we would desire. We can, however, use the cross-ratio to determine whether a given 4-point partition has a conformal center. This analysis yields the following result:

Theorem. *A circle with partition* (z_1, z_2, z_3, z_4) *has a conformal center if and only if*

$$\frac{z_1 z_3 + z_2 z_4}{2} = \left(\frac{z_1 + z_3}{2}\right)\left(\frac{z_2 + z_4}{2}\right).$$

That is, the midpoint of the product of opposite points equals the product of the midpoints of opposite points.

Proof. The circle with partition $(1, i, -1, -i)$ has a conformal center. Therefore, a circle (z_1, z_2, z_3, z_4) has a conformal center if and only if

$$z_1 : z_2 : z_3 : z_4 = 1 : i : -1 : -i$$

That is, we have:

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{(1 - i)(-1 - (-i))}{(1 - (-i))(-1 - i)}$$

The right side evaluates to -1. Expanding the left yields

$$z_1 z_3 - z_2 z_3 - z_1 z_4 + z_2 z_4 = -z_1 z_3 + z_3 z_4 + z_1 z_2 - z_2 z_4$$

Consolidating terms, we arrive at

$$2z_1 z_3 + 2z_2 z_4 = (z_1 + z_3)(z_2 + z_4),$$

which is the desired result. □

Chapter 13

Applications

In a forthcoming paper *The Conformal Center of a Triangle* [5], we discuss how to manipulate the expressions from Chapter 7 to create some new series identities. The series identities we obtain are:

$$\sin \pi a \sin \pi b \Phi(1-a, 1-b) + \sin \pi b \sin \pi c \Phi(1-b, 1-c) + \sin \pi c \sin \pi a \Phi(1-c, 1-a)$$
$$= \frac{\pi}{\sqrt{3}} \left[\cos\left(\frac{2\pi}{3}(b-a)\right) + \cos\left(\frac{2\pi}{3}(c-b)\right) + \cos\left(\frac{2\pi}{3}(a-c)\right) \right]$$

$$(b-a)\Psi(1-a, 1-b) + (c-b)\Psi(1-b, 1-c) + (a-c)\Psi(1-c, 1-a)$$
$$= \pi \left[\sin\left(\frac{2\pi}{3}(b-a)\right) + \sin\left(\frac{2\pi}{3}(c-b)\right) + \sin\left(\frac{2\pi}{3}(a-c)\right) \right]$$

In the special case where $c = 0$, these identities eventually yield the following expressions:

$$1 + \frac{1-t^2}{4 \cdot 2!} + \frac{(1-t^2)(3^2-t^2)}{4^2 \cdot 4!} + \frac{(1-t^2)(3^2-t^2)(5^2-t^2)}{4^3 \cdot 6!} + ... = \frac{2}{\sqrt{3}} \cos\left(\frac{\pi t}{6}\right)$$

$$1 + \frac{1-t^2}{4 \cdot 3!} + \frac{(1-t^2)(3^2-t^2)}{4^2 \cdot 5!} + \frac{(1-t^2)(3^2-t^2)(5^2-t^2)}{4^3 \cdot 7!} + ... = \frac{1}{t} \sin\left(\frac{\pi t}{6}\right)$$

These identities, which can be written as hypergeometric series, are actually related to Chebyshev polynomials. They can be derived by evaluating the Chebyshev polynomials of the first and second kind at $\pi/6$ [9].

Chapter 14

Conclusion

Before we began our research, it was known that a point with the properties of our conformal center existed in a triangle. It was also known that it could be found using harmonic measure and the Schwarz-Christoffel Transformation. Our innovation was to think of this point in the context of geometric triangle centers. We named this point the conformal center, because it is the only triangle center that is invariant under conformal maps.

By analyzing the Schwarz-Christoffel transformation of this point, we found a series expression for its barycentric coordinates. Using the Encyclopedia of Triangle Centers, we compared this point to all known triangle centers, and found that it does not coincide with any of them. It does, however, lie very close the conformal center, as we verified numerically.

Turning to regions other than triangles, we determined a general criterion for finding polygons with conformal centers. In the specific case of quadrilaterals, we were able to find a fairly clean integral expression of the conformal center. In the special case of a symmetric quadrilateral, we found a series expression for the location of the conformal center.

Next, we considered circular domains. We found that every circle with a 3-point partition has a conformal center, and that this point coincides with the First Isodynamic center of the triangle with vertices on these partition points. While a 4-point circle does not generally have a conformal center, we established a simple criterion for finding the special 4-point circles that do. Finally, we discussed the application of this material to evaluating series expressions. We obtained two hypergeometric series identities that appear to be new.

All the series in this paper can be written more concisely as hypergeometric functions, a generalization of geometric series. For simplicity, we have largely avoided their use in this paper, but a reader who is interested in the conformal center should familiarize himself or herself with the basics of hypergeometric series. Both the Wolfram Research website [9] and *Special Functions* [3] contain extensive collections of hypergeometric identities, which are very useful in these analyses. The Appell function, which can be found on Wolfram Research, is a particularly useful hypergeometric series, which we make use of in our forthcoming paper to shorten the derivation in Chapter 7.

It is possible to generalize the concept of a conformal center for quadrilaterals. We saw that a non-square rectangle does not have a conformal center. It does, however, have a unique point from which a random path is equally likely to first exit through either of a pair of opposite sides. That is, in Fig. 2.1, if we call the region bounded by the rectangle Ω, then we have

$$\omega(q, A, \Omega) = \omega(q, C, \Omega),$$

$$\omega(q, B, \Omega) = \omega(q, D, \Omega).$$

Since this point is defined by Brownian probability, or harmonic measure, it is also preserved by conformal maps. Also, although many quadrilaterals lack a conformal center, every quadrilateral has a point like this. That is, it can be shown that every quadrilateral maps conformally onto some rectangle [2]. Only those that map to the square have a conformal center, but even the quadrilaterals that map onto a rectangle with side length ratio $A/B \neq 1$ have the point described here. Using the concepts of modulus and extremal length [2], we can determine the ratio A/B of the rectangle that a given quadrilateral maps to. If I had more time to work on this topic, I would explore this point, and its analogues for higher-order polygons.

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http://functions.wolfram.com/HypergeometricFunctions/

This website has a collection of useful facts and identities, including hypergeometric identities.

http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind
http://mathworld.wolfram.com/ChebyshevPolynomialoftheSecondKind.html

These are descriptions of Chebyshev polynomials of the first and second kind, mentioned in the applications sections.