Cesaro Limits of Analytically Perturbed Stochastic Matrices

Jason Murcko
Advisor: Hank Krieger
Overview

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Motivating example

The peculiar case of Roland the hot dog street vendor
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\[ r_{n+1}(1) = (0.5 + \varepsilon) r_n(1) + (0.5 - \varepsilon) r_n(2) \]
\[ r_{n+1}(2) = (0.5 - 2\varepsilon) r_n(1) + (0.5 + 2\varepsilon) r_n(2) \]
Motivating example

The peculiar case of Roland the hot dog street vendor

\[
\begin{align*}
    r_{n+1}(1) &= (0.5 + \varepsilon)r_n(1) + (0.5 - \varepsilon)r_n(2) \\
    r_{n+1}(2) &= (0.5 - 2\varepsilon)r_n(1) + (0.5 + 2\varepsilon)r_n(2)
\end{align*}
\]

or

\[
\begin{bmatrix}
    r_{n+1}(1) \\
    r_{n+1}(2)
\end{bmatrix} =
\begin{bmatrix}
    0.5 + \varepsilon & 0.5 - \varepsilon \\
    0.5 - 2\varepsilon & 0.5 + 2\varepsilon
\end{bmatrix}
\begin{bmatrix}
    r_n(1) \\
    r_n(2)
\end{bmatrix}
\]
Motivating example (cont.)

The long-term daily average that Roland earns starting at corner 1 is

$$\lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} r_k(1).$$
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\]

From the previous recursive relationship,

\[
\lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \begin{bmatrix} r_k(1) \\ r_k(2) \end{bmatrix} = \lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \begin{bmatrix} 0.5 + \varepsilon & 0.5 - \varepsilon \\ 0.5 - 2\varepsilon & 0.5 + 2\varepsilon \end{bmatrix}^k \begin{bmatrix} r_0(1) \\ r_0(2) \end{bmatrix}
\]
Motivating example (cont.)

\[
P^* = \lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} \begin{bmatrix} 0.5 + \varepsilon & 0.5 - \varepsilon \\ 0.5 - 2\varepsilon & 0.5 + 2\varepsilon \end{bmatrix}^k
\]

\[
= \frac{1}{1 - 3\varepsilon} \begin{bmatrix} 0.5 - 2\varepsilon & 0.5 - \varepsilon \\ 0.5 - 2\varepsilon & 0.5 - \varepsilon \end{bmatrix}
\]
Motivating example (cont.)

\[ P^* = \lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \begin{bmatrix} 0.5 + \epsilon & 0.5 - \epsilon \\ 0.5 - 2\epsilon & 0.5 + 2\epsilon \end{bmatrix}^k \]

\[ = \frac{1}{1 - 3\epsilon} \begin{bmatrix} 0.5 - 2\epsilon & 0.5 - \epsilon \\ 0.5 - 2\epsilon & 0.5 - \epsilon \end{bmatrix} \]

Roland’s long-term average daily earnings are thus

\[ \frac{0.5 - 2\epsilon}{1 - 3\epsilon} \cdot 90 + \frac{0.5 - \epsilon}{1 - 3\epsilon} \cdot 100 = 95 + \frac{5\epsilon}{1 - 3\epsilon} \]
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Segue

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$$\lim_{\varepsilon \downarrow 0} \lim_{N \to \infty} \frac{1}{N + 1} \sum_{k=0}^{N} P^k = \lim_{N \to \infty} \lim_{\varepsilon \downarrow 0} \frac{1}{N + 1} \sum_{k=0}^{N} P^k$$
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What would happen if we let $\varepsilon \downarrow 0$ and $N \to \infty$ simultaneously?
Definition: A square matrix is *stochastic* if all its entries are real and nonnegative and the sum of the entries in each row is equal to 1.
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|λ| \leq 1 for any eigenvalue \(λ\) of a stochastic matrix.

Definition: An *analytic perturbation* of a matrix \(T_0 \in M_n(\mathbb{C})\) is a power series

\[
T(\varepsilon) = T_0 + A(\varepsilon) = T_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots
\]

in which the “coefficients” \(A_1, A_2, \ldots\) are in \(M_n(\mathbb{C})\) as well.
Definition: An analytically perturbed stochastic matrix is an analytic perturbation $P(\varepsilon)$ of a stochastic matrix $P_0$. 
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We want $P(\varepsilon)$ to be stochastic for all sufficiently small positive $\varepsilon$. 
In 2002, Filar, Krieger, and Syed characterized the hybrid Cesaro limit

\[ \lim_{\varepsilon \downarrow 0} \frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} P^k(\varepsilon) \]

for an analytically perturbed stochastic matrix

\[ P(\varepsilon) = P_0 + A(\varepsilon) \]

Subject to the restriction that \( P_0 \) have no eigenvalues \( \lambda \) satisfying \( |\lambda| = 1 \) except for \( \lambda = 1 \).
• What happens if we allow the unperturbed stochastic matrix $P_0$ to have eigenvalues $\lambda \neq 1$ with $|\lambda| = 1$?

• Does the Cesaro limit still necessarily exist?

• If or when the limit does exist, how will such eigenvalues affect the limit?

• How does the rate at which $N(\epsilon) \to \infty$ affect the existence or value of the limit?
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• How does the rate at which $N(\varepsilon) \to \infty$ affect the existence or value of the limit?
If $T(\varepsilon) = T_0 + A(\varepsilon)$ and $\lambda$ is an eigenvalue of $T_0$, then $T(\varepsilon)$ has a collection of eigenvalues $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_s(\varepsilon)$ that converge to $\lambda$ as $\varepsilon \to 0$. Each $\lambda_j(\varepsilon)$ has a Puiseux series $\lambda_j(\varepsilon) = \lambda + c_1^{1/p_j} \varepsilon^{1/p_j} + c_2^{1/p_j} \varepsilon^{2/p_j} + \cdots$ for some positive integer $p_j$ and complex numbers $c_1^{1/p_j}, c_2^{1/p_j}, \ldots$. 
If $T(\varepsilon) = T_0 + A(\varepsilon)$ and $\lambda$ is an eigenvalue of $T_0$, then $T(\varepsilon)$ has a collection of eigenvalues $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_s(\varepsilon)$ that converge to $\lambda$ as $\varepsilon \to 0$.

Each $\lambda_j(\varepsilon)$ has a *Puiseux series*

$$\lambda_j(\varepsilon) = \lambda + c_{1,j}\varepsilon^{1/p_j} + c_{2,j}\varepsilon^{2/p_j} + \cdots$$

for some positive integer $p_j$ and complex numbers $c_{1,j}, c_{2,j}, \ldots$
Results for stochastic matrices

For an analytically perturbed stochastic matrix $P(\varepsilon)$, if $\lambda(\varepsilon) \neq 1$ is a perturbed eigenvalue corresponding to $\lambda = 1$, then the first nonzero coefficient in its Puiseux series has negative real part.
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Perturbed eigenvalues for $\lambda = 1$ cannot approach the unit circle tangentially.

It would be nice to have a similar result for other eigenvalues on the unit circle.
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- All $n$th or less roots of unity
- Curvilinear arcs connecting consecutive roots of unity
Region for $n = 4$