Cesaro Limits of Analytically Perturbed Stochastic Matrices

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Overview

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Motivating example

The peculiar case of Roland the hot dog street vendor
Motivating example

The peculiar case of Roland the hot dog street vendor

\[ p_{n+1}(1) = (0.5 + \varepsilon)p_n(1) + (0.5 - 2\varepsilon)p_n(2) \]
\[ p_{n+1}(2) = (0.5 - \varepsilon)p_n(1) + (0.5 + 2\varepsilon)p_n(2) \]
Motivating example

The peculiar case of Roland the hot dog street vendor

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\[ p_{n+1}(2) = (0.5 - \varepsilon)p_n(1) + (0.5 + 2\varepsilon)p_n(2) \]

or \ldots

\[
\begin{bmatrix}
  p_{n+1}(1) & p_{n+1}(2)
\end{bmatrix} =
\begin{bmatrix}
  p_n(1) & p_n(2)
\end{bmatrix}
\begin{bmatrix}
  0.5 + \varepsilon & 0.5 - \varepsilon \\
  0.5 - 2\varepsilon & 0.5 + 2\varepsilon
\end{bmatrix}
\]
Motivating example (cont.)

The long-term expected portion of the days Roland spends on corner 1 is

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} p_k(1).
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$$

From the previous recursive relationship,

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \begin{bmatrix} p_k(1) & p_k(2) \end{bmatrix} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \begin{bmatrix} 1 & 0 \\ 0.5 + \varepsilon & 0.5 - \varepsilon \\ 0.5 - 2\varepsilon & 0.5 + 2\varepsilon \end{bmatrix}^k
$$
Motivating example (cont.)

\[ P^* = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \begin{bmatrix} 0.5 + \varepsilon & 0.5 - \varepsilon \\ 0.5 - 2\varepsilon & 0.5 + 2\varepsilon \end{bmatrix}^k \]

\[ = \frac{1}{1 - 3\varepsilon} \begin{bmatrix} 0.5 - 2\varepsilon & 0.5 - \varepsilon \\ 0.5 - 2\varepsilon & 0.5 - \varepsilon \end{bmatrix} \]
Motivating example (cont.)

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Roland’s long-term average daily earnings are thus

\[ \frac{0.5 - 2\varepsilon}{1 - 3\varepsilon} \cdot 90 + \frac{0.5 - \varepsilon}{1 - 3\varepsilon} \cdot 100 = 95 + \frac{5\varepsilon}{1 - 3\varepsilon} \]
If we let $\varepsilon \downarrow 0$, we get the amount we would have found if we had let $\varepsilon = 0$ to begin with.
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\[
\lim_{\varepsilon \downarrow 0} \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} P^k = \lim_{N \to \infty} \lim_{\varepsilon \downarrow 0} \frac{1}{N} \sum_{k=1}^{N} P^k
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$$\lim_{\varepsilon \downarrow 0} \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} P^k \quad = \quad \lim_{N \to \infty} \lim_{\varepsilon \downarrow 0} \frac{1}{N} \sum_{k=1}^{N} P^k$$

What would happen if we let $\varepsilon \downarrow 0$ and $N \to \infty$ simultaneously?
Definitions

A square matrix is *stochastic* if all its entries are real and nonnegative and the sum of the entries in each row is equal to 1.
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An *analytic perturbation* of a matrix $T_0 \in \mathbb{C}^{n \times n}$ is a power series

$$T(\varepsilon) = T_0 + A(\varepsilon) = T_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots$$

in which the “coefficients” $A_1, A_2, \ldots$ are in $\mathbb{C}^{n \times n}$ as well.
Putting the Two Together

An *analytically perturbed stochastic matrix* is an analytic perturbation $P(\varepsilon)$ of a stochastic matrix $P_0$. 
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We are interested in the hybrid Cesaro limit

$$\lim_{\varepsilon \downarrow 0} \frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} P_k(\varepsilon),$$

where $N(\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$. 
In 2002, Filar, Krieger, and Syed characterized the hybrid Cesaro limit when $P_0$ has no eigenvalues $\lambda$ satisfying $|\lambda| = 1$ except for $\lambda = 1$.

- Each eigenvalue $\lambda$ of $P_0$ has a separate contribution to the limit.
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- Each eigenvalue $\lambda$ of $P_0$ has a separate contribution to the limit.
- If $|\lambda| < 1$, this contribution is always equal to 0.
- If $\lambda = 1$, the contribution depends on the rate at which $N(\varepsilon) \uparrow \infty$. 
Dependence of Limit on $N(\varepsilon)$

If

$$P(\varepsilon) = \begin{bmatrix} 1 - \varepsilon & \varepsilon \\ \varepsilon & 1 - \varepsilon \end{bmatrix}$$

and $N(\varepsilon) \varepsilon \rightarrow L$ as $\varepsilon \downarrow 0$, where $0 < L < \infty$, 
Dependence of Limit on $N(\varepsilon)$

If

$$P(\varepsilon) = \begin{bmatrix} 1 - \varepsilon & \varepsilon \\ \varepsilon & 1 - \varepsilon \end{bmatrix}$$

and $N(\varepsilon)\varepsilon \to L$ as $\varepsilon \downarrow 0$, where $0 < L < \infty$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} P^k(\varepsilon) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \frac{1 - e^{2L}}{2L} \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$
Perturbed eigenvalues

If $T(\varepsilon) = T_0 + A(\varepsilon)$ and $\lambda$ is an eigenvalue of $T_0$, then $T(\varepsilon)$ has a collection of eigenvalues $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_s(\varepsilon)$ that converge to $\lambda$ as $\varepsilon \to 0$. Each $\lambda_j(\varepsilon)$ has a Puiseux series $\lambda_j(\varepsilon) = \lambda + c_1, j \varepsilon^{1/p_j} + c_2, j \varepsilon^{2/p_j} + \cdots$ for some positive integer $p_j$ and complex numbers $c_1, j, c_2, j, \ldots$. 
If $T(\varepsilon) = T_0 + A(\varepsilon)$ and $\lambda$ is an eigenvalue of $T_0$, then $T(\varepsilon)$ has a collection of eigenvalues $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_s(\varepsilon)$ that converge to $\lambda$ as $\varepsilon \to 0$.

Each $\lambda_j(\varepsilon)$ has a Puiseux series

$$\lambda_j(\varepsilon) = \lambda + c_{1,j}\varepsilon^{1/p_j} + c_{2,j}\varepsilon^{2/p_j} + \cdots$$

for some positive integer $p_j$ and complex numbers $c_{1,j}, c_{2,j}, \ldots$
The reduction process

Given $T(\varepsilon)$, $\lambda$ is *reducible* for $T(\varepsilon)$ if $\lambda$ is a semisimple eigenvalue of $T(0)$. 

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If $\lambda$ is reducible for $T(\varepsilon)$, we can reduce $T(\varepsilon)$ for $\lambda$ to the matrix

$$\tilde{T}(\varepsilon) = \frac{1}{\varepsilon}(T(\varepsilon) - \lambda I)P^*(\lambda, \varepsilon).$$
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$$
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$$

$\lambda$ is completely reducible for $T(\varepsilon)$ if $0$ is reducible for $T_0(\varepsilon) = T(\varepsilon) - \lambda I$ and, inductively, $0$ is reducible for $T_i(\varepsilon)$, where $T_i(\varepsilon)$ is obtained by reducing $T_{i-1}(\varepsilon)$ for $0$. 
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- Curvilinear arcs connecting consecutive roots of unity
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- All \( k \)th roots of unity, where \( k \leq n \)
- Curvilinear arcs connecting consecutive roots of unity
- Each arc implicitly parametrized in \( t \) by one of the following equations:

\[
\begin{align*}
\zeta^q (\zeta^p - t)^r &= (1 - t)^r \\
(\zeta^b - t)^d &= (1 - t)^d \zeta^q
\end{align*}
\]
Region for $n = 4$
• Curvilinear boundary arcs make nonzero angles with lines tangent to the unit circle
Results

- Curvilinear boundary arcs make nonzero angles with lines tangent to the unit circle.
- If $\lambda(\varepsilon)$ is a $\lambda$-group eigenvalue of $P(\varepsilon)$ where $|\lambda| = 1$, then the direction of approach of $\lambda(\varepsilon)$ to $\lambda$ has a nonzero radial component.
Results

• Curvilinear boundary arcs make nonzero angles with lines tangent to the unit circle

• If $\lambda(\varepsilon)$ is a $\lambda$-group eigenvalue of $P(\varepsilon)$ where $|\lambda| = 1$, then the direction of approach of $\lambda(\varepsilon)$ to $\lambda$ has a nonzero radial component.

• If $\lambda \neq 1$ is a completely reducible unit-circle eigenvalue of $P(\varepsilon)$, its contribution to the hybrid Cesaro limit is 0.