Cesaro Limits of Analytically Perturbed Stochastic Matrices

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Markov Chains
The theory of discrete-time Markov chains on a finite state space has a wide array of applications in modeling everything from credit ratings to population genetics. A particular Markov chain can be represented by a stochastic matrix containing the transition probabilities, as follows:

\[ P = \begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.2 & 0.6 & 0.2 \\ 1 & 0 & 0 \end{bmatrix}. \]

Information about the long-term behavior of such a system (that is, the relative frequency it inhabits each state) is encoded in the Cesaro limit:

\[ P^* = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P^k. \]

Imperfect Information and a New Type of Limit
In actual situations, the precise transition probabilities for a Markov chain with perfect information about the long-term behavior of such a system may not be known exactly (for example if they are estimated from data). In this case we may have a sort of transition matrix

\[ P(\varepsilon) = P_0 + A(\varepsilon) \]

where \( P_0 \) is the actual transition matrix for the system and \( A(\varepsilon) \) is an error term that depends upon the parameter \( \varepsilon \). We require that \( P(\varepsilon) \) be stochastic for all sufficiently small positive \( \varepsilon \). For such analytically perturbed stochastic matrices, we can investigate a hybrid Cesaro limit similar to that above:

\[ \lim_{\varepsilon \to 0} \frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} P_k(\varepsilon). \]

Here \( N(\varepsilon) \) increases to \( \infty \) as \( \varepsilon \) decreases to 0. In particular, we are concerned with cases where the unperturbed stochastic matrix \( P_0 \) has eigenvalues on the unit circle in the complex plane other than 1 (in such cases, the associated system exhibits periodic behavior).

Previous Results
In 2002, Filar, Krieger, and Syed characterized this hybrid Cesaro limit when \( P_0 \) has no eigenvalues on the unit circle other than 1. In particular, they showed that the limit always exists, but that it can depend on the rate at which \( N(\varepsilon) \) increases. For example, letting

\[ P(\varepsilon) = \begin{bmatrix} 1 - \varepsilon & \varepsilon \\ \varepsilon & 1 - \varepsilon \end{bmatrix}, \]

then three cases arise:

- If \( N(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \), then the hybrid Cesaro limit is
  \[ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \]
- If \( N(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), the hybrid Cesaro limit is
  \[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]
- And if \( N(\varepsilon) \to \infty \) as \( \varepsilon \to 0 \), where \( 0 < N < \infty \), the limit is
  \[ \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1 - e^{-2L}}{2L} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}. \]

Eigenvalues of Perturbed Matrices
For any analytically perturbed stochastic matrix \( P(\varepsilon) \), its eigenvalues vary continuously with \( \varepsilon \) and hence approach the eigenvalues of \( P_0 \) as \( \varepsilon \to 0 \) (in fact, this is true for all analytically perturbed matrices, not just stochastic ones). For instance, in the example above the eigenvalues of \( P(\varepsilon) \) are

\[ \lambda_1(\varepsilon) = 1, \quad \lambda_2(\varepsilon) = 1 - 2\varepsilon; \]

both of which approach 1, the eigenvalue of \( P_0 \), the identity matrix, as \( \varepsilon \to 0 \).

The eigenvalues of a stochastic matrix, as it turns out, always lie within the closed unit disc in the complex plane. Hence, for an analytically perturbed stochastic matrix, the perturbed eigenvalues will trace out curves within the unit disc as \( \varepsilon \to 0 \).

The results obtained by Filar, Krieger, and Syed relied on showing that for perturbed eigenvalues approaching 1, there is a certain behavior of \( N(\varepsilon) \), that is, the relative frequency it inhabits each state (denoted by \( \theta_\varepsilon \)).

More on Eigenvalues
In 1951, Karpelevic obtained a general characterization of the set of complex numbers that are the eigenvalue of some \( n \times n \) stochastic matrix for each individual \( n \) (this set or region is denoted by \( \Theta_n \)).

The boundary of \( \Theta_n \) consists of all the \( n \)th roots of unity, \( k = 0, \ldots, n - 1 \), together with curved arcs connecting these roots of unity in circular order. Each curved arc is implicitly parametrized by an equation of one of the following forms:

\[ s^n (z^n - 1) = (1 - t)^n \]

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Here the parameter \( t \) is allowed to run over the interval \([0, 1]\), and \( k, d, p, q, \) and \( r \) are nonnegative integers that depend on the endpoints of the arc.

Under an additional assumption that a specific type of reduction process can be performed on \( P(\varepsilon) \), this bounding of the perturbed eigenvalues allows the methods used by Filar, Krieger, and Syed to be adapted and applied to the new situation. In this case, however, the presence of eigenvalues on the unit circle other than 1 adds nothing new, in a sense, to the value of the hybrid Cesaro limit.

New Results
As the diagrams for \( \Theta_n \) suggest, none of the boundary arcs are tangent to the unit circle at their respective endpoints. Using the equations for the arcs and the implicit function theorem, I have been able to show that this is the case for every \( \Theta_n \).

This implies that for an analytically perturbed stochastic matrix \( P(\varepsilon) \), any perturbed eigenvalue that approaches a value on the unit circle as \( \varepsilon \to 0 \) is bounded away from the unit circle in the same way that perturbed eigenvalues approaching 1 were previously know to be; that is, the eigenvalue curves cannot be tangent to the unit circle at their limiting points, the unperturbed eigenvalues.

References