Constructing a Matrix Representation of the Lie Group G_2

Ruben Arenas

Weiqing Gu, Advisor

Vatche Sahakian, Reader

May 9, 2005

Harvey Mudd College
Department of Mathematics

Abstract

We define the Lie group G_2 and show several equivalent ways to view G_2. We do the same with its Lie algebra $\mathfrak{g}_2$. We identify a new basis for $\mathfrak{g}_2$ using Bryant's view of $\mathfrak{g}_2$ and geometric considerations we develop. We then show how to construct a matrix representation of G_2 given our particular basis for $\mathfrak{g}_2$. We examine the geometry of 1- and 2-parameter subgroups of G_2. Finally, we suggest an area of further research using the new geometric characterization we developed for $\mathfrak{g}_2$.

Contents

Abstract **iii**

Acknowledgments **ix**

1 Introduction **1**

2 Background **3**
- 2.1 Lie Groups 3
- 2.2 Lie Algebras 4
- 2.3 The Exponential Map 4
- 2.4 The Lie Group G_2 6

3 Ways of Viewing the Lie Group G_2 **7**
- 3.1 The Octonions 7
- 3.2 G_2 as the Automorphism Group of the Octonions 8
- 3.3 G_2 as Linear Transformations Preserving the Cross Product . 12
- 3.4 G_2 as a Stable Group of an Associative Form 13
- 3.5 G_2 as Linear Maps Preserving the Associator up to a Sign . . 15
- 3.6 Summary 16

4 Ways of Viewing the Lie Algebra $\mathfrak{g}_2$ **19**
- 4.1 Expressing the Associative Calibration 19
- 4.2 The $\epsilon_{\ell mn}$ and Bryant's View of $\mathfrak{g}_2$ 19
- 4.3 Recovering $\mathfrak{g}_2$ from its Dynkin Diagram 20
- 4.4 A Combinatorial Construction of $\mathfrak{g}_2$ 22
- 4.5 Summary 27

5 Results: Relating G_2 and $\mathfrak{g}_2$ **29**
- 5.1 Generating $\mathfrak{g}_2$ using the $\epsilon_{\ell mn}$ 29
- 5.2 Commutativity of Basis Elements 33

5.3 Constructing a Matrix Representation of G_2 35
5.4 Examples 36
5.5 An Area for Future Research: Spin(7) 39

A Exponential of Elements and Commuting Pairs of $\mathbb{S}$ **41**

B A Matrix Representation of G_2 **47**

Bibliography **89**

List of Figures

1.1 Parallel transport on a sphere 2

2.1 All possible Dynkin diagrams 6

3.1 Quaterion multiplication 8
3.2 Octonion multiplication: The Fano Plane 8
3.3 Relationships between the ways of viewing G_2 17

4.1 The Dynkin diagram for G_2. 21
4.2 The root system associated to G_2. 21
4.3 A multiplication table for $\mathfrak{g}_2$ derived from the Dynkin diagram 23
4.4 The G_2-hexagon 24
4.5 Another root system associated to G_2. 25
4.6 Relationships between the ways of viewing $\mathfrak{g}_2$ 28

5.1 Every matrix A in $\mathfrak{g}_2$ is orthogonal to the span of $\{E_1, \ldots, E_7\}$ 31
5.2 Commutation table for $\mathfrak{g}_2$ using $\mathbb{S}$. 34

Acknowledgments

I would like to thank Professor Weiqing Gu for her patience as I began my research. Without her thoughtful guidance throughout the year I certainly could not have produced this thesis.

Many thanks to Professor Vatche Sahakian for his support, both academic and otherwise. Without many insightful conversations with him I would have not been able to complete this work.

Chapter 1

Introduction

The goal of this thesis is to present a new matrix representation of the exceptional Lie group G_2. We begin in Chapter 2 by providing background on Lie groups, Lie algebras, the exponential map, and G_2. Chapter 3 develops the theory of the Lie group G_2 by looking at four major ways from the literature of viewing G_2. These four perspectives require us to develop basic information about the octonions, a generalized cross product, the associative calibration, and the associator. Chapter 4 similarly develops the theory of G_2's Lie algebra, $\mathfrak{g}_2$, by looking at three characterizations. One of these three is Robert Bryant's method which uses the coefficients $\epsilon_{\ell mn}$ of an associative calibration to characterize $\mathfrak{g}_2$. The connection between the associative calibration and $\epsilon_{\ell mn}$ is described in Section 4.1.

In Chapter 5 we build on Bryant's $\epsilon_{\ell mn}$ definition of $\mathfrak{g}_2$ to give a new geometric characterization of $\mathfrak{g}_2$. From this characterization we explicitly construct a basis $\mathbb{S}$ for $\mathfrak{g}_2$ (see Theorem 5.1). This is one of our main results - an explicit construction has not been previously done in the literature. We also present a commutation table for $\mathbb{S}$.

Finally, Chapter 5 also includes information on constructing G_2 using the basis $\mathbb{S}$. Appendix A catalogs the elements of G_2 generated by single elements of $\mathbb{S}$ and commuting pairs of $\mathbb{S}$. We examine these in Chapter 5 and talk about the geometry of G_2. We also carry out several examples using these elements of G_2 by showing how they satisfy the conditions imposed by Chapter 2. For completeness, Appendix B gives the explicit 14-parameter matrix representation of G_2.

In summary, the major original results of this work are

- A geometric construction of $\mathfrak{g}_2$ (Section 5.1).

- The elements of G_2 corresponding to elements of $\mathbb{S}$ under the exponential map (Appendix A).
- A complete description of G_2 in 14 parameters (Appendix B).

Recent motivation for this research comes from physics. In particular, M-theory suggests that the universe has the structure of $\mathbb{R}^{3,1} \times M^7$ where M^7 is a manifold with G_2-holonomy. This means that the holonomy group of M^7 is G_2. By the holonomy group of a connected Riemannian manifold we mean the group of transformations that are induced by parallel transporting all possible tangent vectors along all possible closed curves through some chosen point. For example, we suspect that the holonomy group for sphere is the set of all rotations in 3-space (see Figure 1.1). The

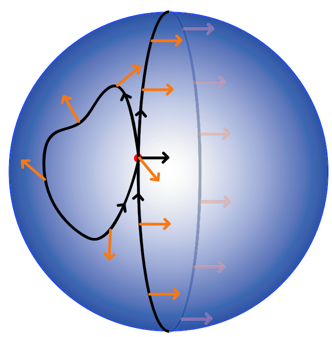

Figure 1.1: Parallel transport on a sphere

holonomy group naturally gives information about the curvature of a manifold as well as other geometric symmetry properties. Thus understanding G_2 should in turn tell us something about the symmetries of M^7. To be precise, subgroups of G_2, such as the 1- and 2-parameter families we describe in Section 5.3, may help us understand certain submanifolds of M^7. Thus having an explicit and complete description of G_2 is a valuable resource for physicists.

The geometric characterization of $\mathfrak{g}_2$ we develop in this work is a novel result. This result is important to Lie theory because it gives an entirely new way of looking at Lie groups that preserve a calibration. In Section 5.5, we describe how our results may be applied in further research.

Chapter 2

Background

Most of the material in this chapter is drawn from basic texts in Lie theory such as [11; 12; 13].

2.1 Lie Groups

In simplest terms, a Lie group is a differentiable manifold endowed with a group structure. To be precise, if G is a differentiable manifold, then G is a *Lie group* if it is also a group, and the multiplication map $m(g, h) = gh$ and the inversion map $i(g) = g^{-1}$ are differentiable. The most trivial example of a Lie group is $\mathbb{R}^n$ under vector addition. Under multiplication, $\mathbb{R}^*$ is a one-dimensional Lie group and $\mathbb{C}^*$ is a 2-dimensional (real) Lie group.

We define the *circle group* S^1 by $\{z \in \mathbb{C} | \ \|z\| = 1\}$. This is a 1-dimensional abelian Lie group. It is easily shown that direct product of Lie groups is also Lie groups. The *n-torus* $T^n = S^1 \times \cdots \times S^1$ is an n-dimensional Lie group.

The *general linear group*, $\mathrm{GL}(n)$, is the group of n-by-n invertible matrices with real entries. This forms an n^2-dimensional Lie group where the coordinate maps are defined by the entries of the matrices. The subgroup of orthogonal matrices $\mathrm{O}(n)$ of $\mathrm{GL}(n)$ is also a Lie group. Our focus will be on *matrix Lie groups*, Lie groups lying in $\mathrm{GL}(n)$.

Since Lie groups are differentiable manifolds, and of course, topological spaces, topological properties apply to Lie groups. For example, $\mathrm{O}(n)$ is a manifold of 2 components (distinguished by positive and negative determinant), while $\mathrm{SO}(n)$, orthogonal matrices with positive determinant, is a connected space. Further, the Lie group of invertible matrices with unit determinant, $\mathrm{SL}(n)$, is closed since it is the inverse image of $\{1\}$ under det, a continuous map. An important property of connected Lie groups is that every open set of the identity generates the entire Lie group. Thus all there is to know about a connected Lie group is encoded near the identity.

2.2 Lie Algebras

There are several different ways to define a Lie algebra. Since we will only be concerned with Lie algebras associated with Lie groups, we define a Lie algebra in terms of a Lie group. If G is a Lie group, then the *Lie algebra* $\mathfrak{g}$ of a Lie group G is the tangent space at the identity of G. The algebra structure on $\mathfrak{g}$ is given by the bracket $[\cdot,\cdot]$ operator. For matrix Lie groups, the bracket is defined by the *commutator*, $[X, Y] = XY - YX$.

As an example, consider $\mathrm{O}(n)$. Let $A(t)$ be any curve in $\mathrm{O}(n)$ with $A(0) = I$. Then the tangent vector to A, A', is a general element of $\mathfrak{o}(n)$. Now since $A^T A = I$, we have by differentiating,

$$A^T A' + A(A^T)' = 0.$$

Evaluating at $t = 0$, and noting that $(A^T)' = (A')^T$, we find that $A' \in \mathfrak{o}(n)$ if and only if

$$A' = -(A')^T.$$

That is, the elements of the Lie algebra $\mathfrak{o}(n)$ are skew-symmetric matrices. Incidentally, $\mathfrak{o}(n) = \mathfrak{so}(n)$, and it is common practice to denote the Lie algebra of $\mathrm{O}(n)$ by $\mathfrak{so}(n)$.

2.3 The Exponential Map

A Lie algebra as we saw previously is geometrically related to a Lie group. So a natural question at this point is if there is an easy way to move from elements in a Lie algebra to elements in a Lie group. And if so, just how much information does the Lie algebra encode about the Lie group? To answer these questions, we introduce the *matrix exponential*

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

for A in $\mathfrak{gl}(n)$. We have the following facts about the exponential.

Proposition 2.1. 1. *The exponential map is a differentiable map from $\mathfrak{g}$ to G.*

2. *The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ onto some neighborhood of the identity in G.*

3. *For any A and B in $\mathfrak{g}$, $\exp(A+B) = \exp(A)+\exp(B)$ if and only if $[A,B]=0$.*

The surprising result of this proposition is that there is an open neighborhood of the identity in G that is diffeomorphic to an open set about 0 of the Lie algebra. Since any open set about the identity generates all of connected G, it happens that $\mathfrak{g}$, a linear structure, encodes all of G. This proposition gives the most common way of associating an element of a Lie algebra $\mathfrak{g}$ to one of G. Fix a basis $\{A_1,\dots,A_n\}$ of $\mathfrak{g}$. Then if $a_1A_1+\cdots+a_nA_n$ is an element of $\mathfrak{g}$,

$$(a_1,\dots,a_n)\mapsto \exp(a_1A_1+\cdots+a_nA_n)$$

is a diffeomorphism onto some open set about the identity in G. This coordinate system is called the *canonical coordinates of the first kind*.

There is a second major way to generate a Lie group from a Lie algebra.

Proposition 2.2. *Let $\mathfrak{g}=\mathfrak{a}_1\oplus\cdots\oplus\mathfrak{a}_n$ be a decomposition of a Lie algebra $\mathfrak{g}$ as a direct sum of subspaces. Then the map*

$$\xi_1+\cdots+\xi_n\mapsto \exp(\xi_1)\cdots\exp(\xi_n)\quad(\xi_i\in\mathfrak{a}_i)$$

maps some neighborhood of 0 in $\mathfrak{g}$ diffeomorphically onto a neighborhood of the identity in G.

For $a_1A_1+\cdots+a_nA_n$ in $\mathfrak{g}$,

$$(a_1,\dots,a_n)\mapsto \exp(a_1A_1)\cdots\exp(a_nA_n)$$

are the *canonical coordinates of the second kind*.

As an example, consider $\mathfrak{so}(3)$. A basis for $\mathfrak{so}(3)$ is the set

$$\left\{\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\right\}.$$

The canonical coordinates of the second kind in this case are $R=(R_1,R_2,R_3)$ where

$$R_1=\begin{pmatrix}\cos(a_1)\cos(a_2)\\-\sin(a_1)\cos(a_2)\\-\sin(a_2)\end{pmatrix}\ R_2=\begin{pmatrix}\sin(a_1)\cos(a_3)-\cos(a_1)\sin(a_2)\sin(a_3)\\\cos(a_1)\cos(a_3)+\sin(a_1)\sin(a_2)\sin(a_3)\\-\cos(a_2)\sin(a_3)\end{pmatrix}$$

$$R_3=\begin{pmatrix}\sin(a_1)\sin(a_3)+\cos(a_1)\sin(a_2)\cos(a_3)\\\cos(a_1)\sin(a_3)-\sin(a_1)\sin(a_2)\cos(a_3)\\\cos(a_2)\cos(a_3)\end{pmatrix}$$

Note that this corresponds to a composition of rotations about three orthogonal axes as we would expect.

2.4 The Lie Group G_2

Cartan classified the semi-simple finite dimensional complex Lie algebras. The idea is that to each Lie algebra there is exactly one Dynkin diagram. Dynkin diagrams correspond to geometric objects called root systems. Root systems are easily classified, and thus all Lie algebras are classified [11]. We list the Dynkin diagrams in Figure 2.1. We have four infinite families of Lie

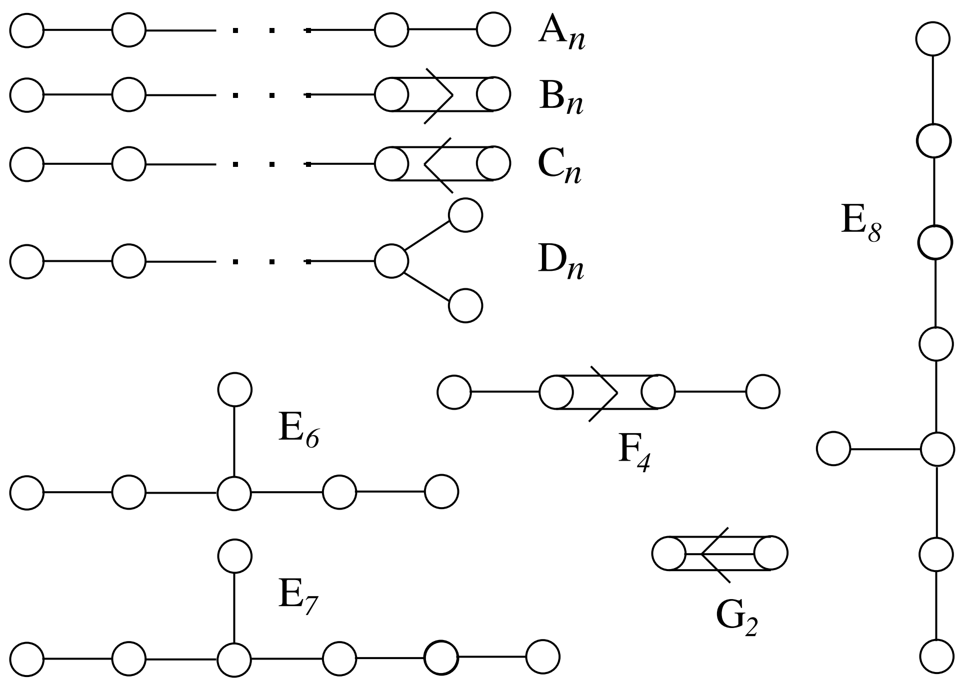

Figure 2.1: All possible Dynkin diagrams

algebras,

$$A_n \iff \mathfrak{sl}_{n+1}\mathbb{C}, B_n \iff \mathfrak{so}_{2n+1}\mathbb{C}, C_n \iff \mathfrak{sp}_{2n}\mathbb{C}, D_n \iff \mathfrak{so}_{2n}\mathbb{C},$$

and five "exceptional" Lie algebras,

$$E_6 \iff \mathfrak{e}_6, E_7 \iff \mathfrak{e}_7, E_8 \iff \mathfrak{e}_8, F_4 \iff \mathfrak{f}_4, \text{ and } G_2 \iff \mathfrak{g}_2.$$

Now $\mathfrak{g}_2$ is the Lie algebra corresponding to the Lie group G_2. It can be shown that G_2 must be 14-dimensional, compact, connected, simply connected, and simple.

Chapter 3

Ways of Viewing the Lie Group G_2

The purpose of this chapter is to develop an understanding of G_2 from several different points of view.

3.1 The Octonions

One of the keys to our discussion of G_2 is the algebraic structure known as the octonions. These can be thought of as a generalization of the quaterions, which are in turn a generalization of the complex numbers. We can describe the quaterions, denoted $\mathbb{H}$, as a 4-dimensional algebra with basis $1, i, j, k$ where 1 is the multiplicative identity, $i^2 = j^2 = k^2 = -1$, and multiplication of i, j, and k is cyclic, as pictured in Figure 3.1, where if we move with an arrow, then the result of the product of the previous two elements is the next element in the cycle, and if we move against an arrow, then it is negative the next element in the cycle. For example, $ki = j$ while $ik = -j$. Not that the quaterions are not commutative.

We similarly define the octonions, denoted $\mathbb{O}$, as an 8-dimensional algebra with basis $1, e_1, \ldots, e_7$. Here, 1 is the identity and $e_1^2 = \cdots = e_7^2 = -1$. We can draw a diagram similar to Figure 3.1 that describes the multiplication of the imaginary basis of $\mathbb{O}$ called the *Fano plane*, pictured in Figure 3.2. The result of the multiplication $e_\ell e_m$ for $\ell \neq m$ is $\pm e_n$, where e_n is the third element lying on the circle or line containing e_ℓ and e_m. The sign is determined according to the direction of the arrow from e_m to e_n. Note how it is possible to relabel the e_ℓ so that the quaterions become an obvious

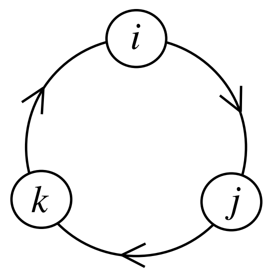

Figure 3.1: Quaterion multiplication

subalgebra of the octonions. If we take $e_4 = e$, then

$$e_1 = i, e_2 = j, e_3 = k, e_4 = e, e_5 = ie, e_6 = je, e_7 = ke,$$

that is $\mathbb{H} = \mathbb{O} \oplus (e\mathbb{O})$. We see from the Fano plane that the octonions are neither commutative nor associative. We will speak more of the octonions

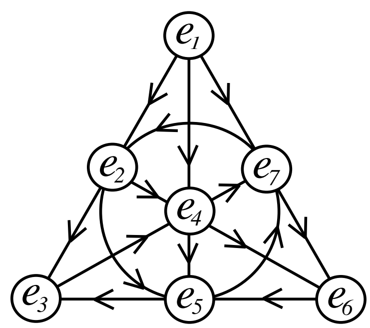

Figure 3.2: Octonion multiplication: The Fano Plane

as a normed division algebra in the next section. For a informative survey of the octonions and their applications see [3].

3.2 G_2 as the Automorphism Group of the Octonions

There are many ways to look at G_2. The challenge for us is to choose a simple way which we will take as our definition of G_2. This definition should be both easy to understand and nearly effortless to work with. Perhaps the simplest way of viewing G_2 is as an automorphism group. With this goal in mind we take a detour to examine the normed division algebras.

3.2.1 Normed Division Algebras

The material in this section is meant as a brief introduction to a vast subject. I am essentially summarizing the results of [14], [15], and [18] and recommend you see those works for a more detailed treatment.

By a *normed algebra* we mean a finite dimensional real algebra $\mathbb{A}$ with identity 1, inner product $\langle \cdot, \cdot \rangle$, and norm $\|\cdot\|$ that satisfies

$$\|xy\| = \|x\|\|y\| \quad \text{for all } x, y \in \mathbb{A}.$$

Further, the norm is connected to the inner product by the relationship

$$\|x\| = \langle x, x \rangle \quad \text{for all } x \in \mathbb{A}.$$

Using this relationship we can determine the inner product from the norm using polarization,

$$\|x + y\| = \langle x + y, x + y \rangle = \|x\| + 2\langle x, y \rangle + \|y\|$$

so that

$$\langle x, y \rangle = \frac{1}{2} \left(\|x + y\| - \|x\| - \|y\| \right).$$

We will be dealing with normed division algebras specifically - algebras where every element but the additive identity has a multiplicative inverse.

What are examples of normed division algebras? It is easy to check that $\mathbb{R}$ with the usual inner product is a normed division algebra. Further, the close cousin of the real numbers, the complex numbers $\mathbb{C}$ with inner product $\langle z, w \rangle = \bar{z}w$, is also a normed division algebra.

Seeing how $\mathbb{C}$ has a notion of *conjugation*, it may be useful for us to have a concept of a *conjugate* defined on a normed division algebra in general. We recall that for the complex numbers the conjugate of $z = a + bi$ is defined by $\bar{z} = a - bi$. Using this as a starting point, we define the real part of a normed algebra $\mathbb{A}$ as

$$\text{Re } \mathbb{A} = \text{span } \{1\}.$$

We define the imaginary part of A as everything but the real part. That is, $\text{Im } \mathbb{A}$ is the orthogonal complement of $\text{Re } \mathbb{A}$. It follows that every element x in $\mathbb{A}$ can be decomposed uniquely into a piece that lies in $\text{Re } \mathbb{A}$ plus a piece that lies in $\text{Im } \mathbb{A}$. That is,

$$x = x_1 + x' \quad \text{where } x_1 \in \text{Re } \mathbb{A} \text{ and } x' \in \text{Im } \mathbb{A}.$$

Now, mimicking the complex conjugate, it is easy to define the conjugate of x as

$$\overline{x} = x_1 - x'.$$

There are many useful properties of norms, inner products, and conjugates, some of which we summarize in the following proposition.

Proposition 3.1. *For a normed division algebra $\mathbb{A}$, the following are equivalent*

1. $\|xy\| = \|x\|\|y\|$
2. $\langle xw, yw\rangle = \langle x, y\rangle\|w\|$
3. $\langle wx, wy\rangle = \|w\|\langle x, y\rangle$
4. $\langle xz, yw\rangle + \langle yz, xw\rangle = 2\langle x, y\rangle\langle z, w\rangle.$

Furthermore, the following properties hold,

1. $\langle xy, z\rangle = \langle x, z\overline{y}\rangle$
2. $\overline{\overline{x}} = x$ *and* $\langle \overline{x}, \overline{y}\rangle = \langle x, y\rangle$
3. $\langle x, y\rangle = \mathrm{Re}\, x\overline{y} = \mathrm{Re}\, \overline{x}y$
4. $\overline{xy} = \overline{yx}$
5. $x\overline{x} = \overline{x}x = \|x\|.$

Recall that $\mathbb{C}$ can be thought of as two copies of $\mathbb{R}$ with multiplication defined by

$$(a + bi)(c + di) = (ac - db) + (ad + cb)i,$$

where i is the imaginary unit. At this point you might ask if it is possible to take two copies of $\mathbb{C}$ to build another normed division algebra. The answer turns out to be "yes". We find that $\mathbb{C} \oplus \mathbb{C}$ is $\mathbb{H}$, another normed division algebra with the Hermitian symmetric norm. Further, $\mathbb{H} \oplus \mathbb{H} = \mathbb{O}$, yet another normed division algebra with Hermitian symmetric norm. This suggests for a normed division algebra $\mathbb{A}$ we can define multiplication and conjugation on $\mathbb{A} \oplus \mathbb{A}$ by

$$\begin{aligned}(a, b)(c, d) &= (ac - d\overline{b}, \overline{a}d + cb)\\ \overline{(a, b)} &= (\overline{a}, -b).\end{aligned}$$

We call this algorithm of "complexification" the *Cayley-Dickson process*.

In each step of "complexification" the resulting normed division algebra loses key properties. In going from $\mathbb{R}$ to $\mathbb{C}$ we lose ordering, from $\mathbb{C}$ to $\mathbb{H}$ we lose commutativity, $\mathbb{O}$ is not associative, and $\mathbb{O} \oplus \mathbb{O}$ fails to even be normed. These observations lead to an astounding theorem by Hurwitz.

Theorem 3.1. (Hurwitz) *Up to isomorphism, the only real normed division algebras are* $\mathbb{R}, \mathbb{C}, \mathbb{H}$, *and* $\mathbb{O}$.

For a proof of Hurwitz's theorem see [14].

3.2.2 A Definition of G_2

With the octonions and their properties nailed down we can now give a convenient definition of G_2.

Definition 3.1. We define G_2 as the group of automorphisms of the octonions. That is,

$$G_2 = \mathrm{Aut}(\mathbb{O}) = \{g \in \mathrm{GL}(\mathbb{O}) \cong \mathrm{GL}_8(\mathbb{R}) \mid g(xy) = g(x)g(y) \text{ for all } x, y \in \mathbb{O}\}.$$

In [14], Harvey shows that for any normed division algebra $\mathbb{A}$ that the automorphisms of $\mathbb{A}$ lie in the orthogonal group on the imaginary part of $\mathbb{A}$. It follows that

$$G_2 \subseteq \mathrm{O}(\mathrm{Im}\,\mathbb{O}) \cong \mathrm{O}(\mathbb{R}^7)$$

which will be important to us when we begin looking for matrix representations of G_2.

We note for completeness that there is a general theory that relates the four normed division algebras to Lie groups. In this framework, a subset of the Lie groups arise as "twisted" isomorphisms of the normed division algebras. For example, Spin(7) is the group of twisted isomorphisms of the octonions and G_2 is the group of special twisted isomorphisms of the octonions. It just happens that the group of special twisted isomorphisms of $\mathbb{O}$ corresponds with $\mathrm{Aut}(\mathbb{O})$. For more on this see [18].

At this point we have not actually shown that G_2 is a Lie group, and in particular the correct Lie group. To show this, we will need a slightly different definition of G_2 which we will find to be equivalent to the automorphism definition. In order to prove this equivalence, we need the *cross product*, which yields a G_2 characterization of its own.

3.3 G_2 as Linear Transformations Preserving the Cross Product

In this section we generalize the notion of a cross product to $\mathbb{O} \cong \mathbb{R}^8$.

Definition 3.2. For all x and y in $\mathbb{O}$ we define the *cross product* of x and y by

$$x \times y = \frac{1}{2}(\overline{y}x - \overline{x}y) = \text{Im }(\overline{y}x). \tag{3.1}$$

Equation (3.1) is an appropriate generalization of the cross product because $x \times y$ is alternating and $\|x \times y\| = \|x \wedge y\|$. We are especially interested in the action of the cross product on the imaginary octonions since G_2 acts on those. It is easy to check that if x and y are in $\text{Im }\mathbb{O} \cong \mathbb{R}^7$ then

$$x \times y = \frac{1}{2}[x, y] = xy + \langle x, y \rangle$$

where $[x, y]$ is the commutator of x and y. We can further restrict ourselves to x and y in $\text{Im }\mathbb{H} \cong \mathbb{R}^3 \subset \text{Im }\mathbb{O}$. In this case $x \times y$ corresponds to the usual cross product on $\mathbb{R}^3$ as we should hope.

We can now show that preserving the cross product is equivalent to being an element of G_2.

Proposition 3.2. *G_2 is precisely the group that preserves the cross product. That is,*

$$G_2 = \{g \in \text{O}(\text{Im }\mathbb{O}) \mid g(x \times y) = g(x) \times g(y) \textit{ for all } x, y \in \text{Im }\mathbb{O}\}.$$

Proof. Suppose first that $g \in G_2 \subset \text{O}(7)$. Note that for all x and y in $\text{Im }\mathbb{O}$ that $g(\langle x, y \rangle) = \langle g(x), g(y) \rangle$ because $\langle x, y \rangle = \overline{x}y$ and $g(\overline{x}) = \overline{g(x)}$ for an isometry g. Thus

$$\begin{aligned} g(x \times y) &= g(xy + \langle x, y \rangle) = g(xy) + g(\langle x, y \rangle) \\ &= g(x)g(y) + \langle g(x), g(y) \rangle = g(x) \times g(y). \end{aligned}$$

as desired.

Now take $g(x \times y) = g(x) \times g(y)$ for all g in $\text{O}(\text{Im }\mathbb{O})$. Then

$$\begin{aligned} g(x \times y) &= g(xy + \langle x, y \rangle) = g(xy) + g(\langle x, y \rangle) \\ &= g(xy) + \langle g(x), g(y) \rangle \end{aligned}$$

because g is an isometry. Yet

$$g(x) \times g(y) = g(x)g(y) + \langle g(x), g(y) \rangle$$

so that

$$g(xy) = g(x)g(y).$$

Therefore $g \in G_2$ and $G_2 = \{g \in \mathrm{O}(\mathrm{Im}\, \mathbb{O}) \mid g(x \times y) = g(x) \times g(y)\}$. □

3.4 G_2 as a Stable Group of an Associative Form

3.4.1 The Associative Calibration ϕ

In this section we focus on the trilinear form $\phi(x, y, z) = \langle x, yz \rangle$. In particular, we are concerned with the action of this form on $\mathrm{Im}\, \mathbb{O} \cong \mathbb{R}^7$. In this case we can apply Proposition 3.1 to see

$$\phi(x, x, z) = \langle x, xz \rangle = \|x\| \langle 1, z \rangle = 0 \tag{3.2}$$

since $z \in \mathrm{Im}\, \mathbb{O}$ and $1 \perp \mathrm{Im}\, \mathbb{O}$. We can show in a similar way that $\phi(x, y, x) = \phi(x, y, y) = 0$ so that ϕ is an *alternating* trilinear form. Therefore we can express ϕ in terms of the basis $\{e^0, e^1, \ldots, e^7\}$ dual to $\{1, e_1, \ldots, e_7\}$. If we let $e^{\ell m n} = e^\ell \wedge e^m \wedge e^n$ then

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$

This, of course, is only one of many ways we could have expressed ϕ. We explore in Chapter 4 the different ways in which ϕ can be written as a result of symmetry.

There is more structure to the form ϕ than is immediately evident. First, let us recall that an alternating p-form can be viewed as an oriented p-plane. That is, each element of $\bigwedge^p V$ corresponds to an oriented p-dimensional subspace (hyperplane) in V. We denote the set of all oriented p-dimensional subspaces of V by $G(p, V)$; called the *grassmannian of oriented p-dimensional subspaces of V*. Since any 3-dimensional subspace of $\mathrm{Im}\, \mathbb{O}$ is isomorphic to the imaginary part of the quaterion subalgebra $\mathbb{H}$, we see ϕ acts on 3-planes in $\mathrm{Im}\, \mathbb{O}$. Said another way, ϕ acts on $G(3, \mathrm{Im}\, \mathbb{O})$. This leads us to the following definitions which we take from [15].

Definition 3.3. If $\omega \in G(3, \mathrm{Im}\, \mathbb{O})$ is the canonically oriented imaginary part of any quaternion subalgebra of $\mathbb{O}$, then the oriented 3-plane is called *associative*. The set of all associative elements of $G(3, \mathrm{Im}\, \mathbb{O})$ is denoted $G(\phi)$ and is called the *associative grassmannian*. We call the 3-form ϕ defined in equation (3.2) the *associative calibration on* $\mathrm{Im}\, \mathbb{O}$.

The idea of a calibration will not be explored further in this paper. For more on ϕ as a calibration see [15; 14].

3.4.2 Redefining G_2

We imagine that some relationship between ϕ and G_2 must exist since both act on the imaginary octonions. The beginning of this relationship is seen in the following proposition.

Proposition 3.3. *If $g \in G_2$ then g fixes the associative calibration.*

This statement is easily proved in general for an associative calibration on an arbitrary space V. Note that for $g \in G_2$

$$\phi(g(x), g(y), g(z)) = \langle g(x), g(y)g(z)\rangle = \langle g(x), g(yz)\rangle = \langle x, yz\rangle = \phi(x, y, z)$$

since $g \in G_2 \subset \mathrm{O}(7)$ is an orthogonal transformation.

Hence, if g is in G_2 then $g^*\phi = \phi$. If the converse were to hold then G_2 is precisely the group of transformations that preserves the associative calibration. In fact, a result due to Bryant [5] shows that the converse does hold, but not only that, he also shows that G_2 is precisely the group we want it to be.

Theorem 3.2. (Bryant) *The subgroup $\{g \in GL(\mathrm{Im}\,\mathbb{O}) \mid g^*\phi = \phi\}$ is a compact, connected, simple, simply connected Lie group of dimension 14. Moreover, this subgroup is isomorphic to $G_2 = \mathrm{Aut}\,\mathbb{O}$.*

We have now confirmed that G_2 is in fact a Lie group, and specifically the Lie group we wanted. Beyond this, there equivalent ways of viewing G_2 are available to us, each of which may be suited to one type of application or another. In the next section we introduce the *coassociative calibration* with the purpose of narrowing a possible supergroup of G_2 further from $\mathrm{O}(7)$ to $\mathrm{SO}(7)$.

3.4.3 The Coassociative Calibration

Define a 4-form ψ on $\mathrm{Im}\,\mathbb{O}$ by

$$\psi(x, y, z, w) = \frac{1}{2}\langle x, y(\bar{z}w) - w(\bar{z}y)\rangle.$$

Note that $\psi(x, y, z, w) = 0$ when any two of x, y, w, or z are set equal so that $\psi \in \bigwedge^4(\mathrm{Im}\,\mathbb{O})^*$. We call ψ the *coassociative calibration on* $\mathrm{Im}\,\mathbb{O}$. You can check that G_2 fixes the coassociative calibration.

The coassociative calibration can be written in terms of a basis for Im $\mathbb{O}$ just as ϕ was. We find,

$$\psi = e^{4567} - e^{4523} - e^{4163} - e^{4127} + e^{2367} + e^{1357} + e^{1256}.$$

But direct computation shows $\phi \wedge \psi = 7e^{1234567}$ so that

$$\frac{1}{7}\phi \wedge \psi$$

is a volume form on Im $\mathbb{O}$. This volume form induces an orientation on Im $\mathbb{O}$ so that the Hodge star operator is well-defined and we can calculate

$$\psi = \star\phi.$$

This information allows us to further restrict the group in which G_2 lies.

Proposition 3.4. $G_2 \subseteq \mathrm{SO}(\mathrm{Im}\,\mathbb{O}) \cong \mathrm{SO}(7)$.

Proof. Now since $g \in G_2$ fixes ϕ and ψ

$$g^*\left(\frac{1}{7}\phi \wedge \psi\right) = \frac{1}{7}\phi \wedge \psi$$

so that g also fixes our chosen volume form. But

$$g^*\left(\frac{1}{7}\phi \wedge \psi\right) = \det(g)\left(\frac{1}{7}\phi \wedge \psi\right) = \frac{1}{7}\phi \wedge \psi$$

and we see that $\det(g) = 1$. Thus G_2 lies in SO(7). □

3.5 G_2 as Linear Maps Preserving the Associator up to a Sign

The associator is the natural "generalization" of the commutator. It measures the associativity of elements in an algebra. Specifically, for all x, y, z,

$$[x, y, z] = (xy)z - x(yz).$$

Since $\mathbb{O}$ is not associative, the associator will not in general vanish for octonion numbers.

We require the following couple of propositions from Harvey and Lawson [15] before we continue.

Proposition 3.5. *Let $\zeta \in G(3,7)$ be an oriented 3-plane. Then $\phi(\zeta) = 1$ if and only if ζ is associative.*

Proposition 3.6. *Take ζ to be an oriented 3-plane in $\mathrm{Im}\,\mathbb{O}$. Then either ζ or $-\zeta$ is associative if and only if $[x,y,z] = 0$ for $\zeta = x \wedge y \wedge z$.*

With these in hand we can give yet another characterization of G_2.

Proposition 3.7. *Let $g \in \mathrm{Im}\,\mathbb{O}$. Then g preserves the associator if and only if g or $-g$ is in G_2.*

Proof. Suppose first that g fixes the associator. Let $[x,y,z] = 0$ with $\zeta = x \wedge y \wedge z$. Then $[g(x), g(y), g(z)] = 0$ and $g(x) \wedge g(y) \wedge g(z)$ is associative. By Proposition 3.5,

$$\phi(g(x) \wedge g(y) \wedge g(z)) = g^*\phi(x \wedge y \wedge z) = 1.$$

But ζ is associative so $\phi(\zeta) = 1$ and

$$g^*\phi = \phi.$$

A similar case holds for $-g$.

Now take $g \in G_2$. Then for any associative $\zeta = x \wedge y \wedge z$, $\phi(\zeta) = 1$. But then,

$$g^*\phi(\zeta) = \phi(g(x) \wedge g(y) \wedge g(z)) = \phi(\zeta) = 1$$

so that $g(x) \wedge g(y) \wedge g(z)$ is also associative. Further,

$$[g(x), g(y), g(z)] = 0 = [x,y,z].$$

Again, this argument remains the same for $-g$. Thus the group that fixes the associator is precisely $G_2 \times \mathbb{Z}_2$. □

3.6 Summary

In Figure 3.3 we summarize the different ways we can view G_2. Arrows between boxes represent an equivalence proved in the literature. For a discussion of G_2 as the subgroup of Spin(7) that fixes an element of S^7 see [2; 18; 16]. For isomorphism concerning G_2, such as $G_2/\mathrm{SU}(3) \cong S^6$, see [15; 16]. Finally, for information on $G(\varphi) = G_2/\mathrm{SO}(4)$ [15].

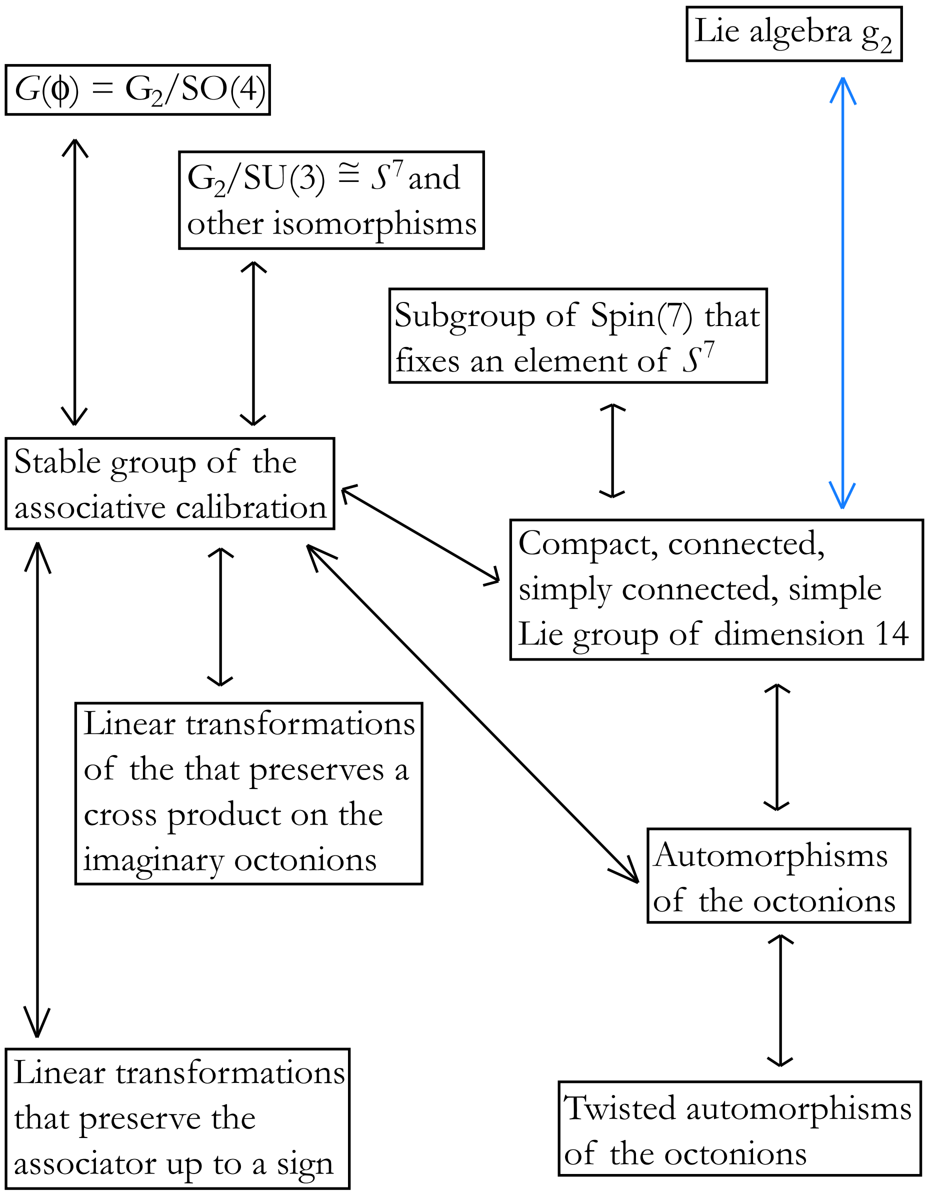

Figure 3.3: Relationships between the ways of viewing G_2

Chapter 4

Ways of Viewing the Lie Algebra $\mathfrak{g}_2$

4.1 Expressing the Associative Calibration

As noted earlier, the associative calibration can be written in terms of the dual basis in many different ways. Consider

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$

By the antisymmetry of the wedge product we can also write

$$\phi = e^{123} - e^{154} - e^{176} - e^{264} - e^{257} - e^{347} - e^{356}$$

In fact, there are $(3!)^7$ ways of expressing the associative calibration. It will be convenient for us to choose one way as preferred. We follow the approach of Bryant [6] and introduce the symbol $\epsilon_{\ell mn}$ defined by $e_\ell \times e_m = \epsilon_{\ell mn} e_n$. For example, since $e_1 \times e_7 = \mathrm{Im}(\bar{e}_7 e_1) = e_6$, we see $\epsilon_{176} = 1$ and $\epsilon_{17n} = 0$ of $n \neq 6$. Using this symbol, we have

$$\phi = \frac{1}{6}\epsilon_{\ell mn} e^{\ell mn}$$

where summation over repeated indices is implied.

4.2 The $\epsilon_{\ell mn}$ and Bryant's View of $\mathfrak{g}_2$

Now Bryant [6] points out a pair of characterizations of $\mathfrak{g}_2$ using the ϵ symbol. Recall that since G_2 is a subgroup of $\mathrm{SO}(7)$ then the Lie algebra $\mathfrak{g}_2$ is a subalgebra of $\mathfrak{so}(7)$, the space of 7×7 skew-symmetric matrices.

Proposition 4.1. *If $a = (a_{\ell m}) \in \mathfrak{so}(7)$, then a is in $\mathfrak{g}_2$ if and only if $\epsilon_{\ell mn} a_{mn} = 0$ for all ℓ.*

Although this way of viewing $\mathfrak{g}_2$ does not immediately suggest a simple way of generating the elements of $\mathfrak{g}_2$, it does provide us with a quick way to check if a given skew-symmetric matrix lies in $\mathfrak{g}_2$. In fact, if we change our notation slightly, Proposition 4.1 does indeed provide a fruitful method of generating $\mathfrak{g}_2$ which we explore in the Chapter 5.1.

Using the ϵ symbols we generate yet another way of looking at $\mathfrak{g}_2$. For $v = (v_1, \ldots, v_n) \in \mathbb{R}^7$ we define a map $[\cdot] : \mathbb{R}^7 \to \mathfrak{so}(7)$ by

$$[v] = (v_{\ell m}) \quad \text{where} \quad v_{\ell m} = \epsilon_{\ell mn} v_n.$$

The matrix $[v]$ is simply the matrix representation of v acting some vector by the cross product. We obtain an irreducible G_2-invariant decomposition of $\mathfrak{so}(7)$ as

$$\mathfrak{so}(7) = \mathfrak{g}_2 \oplus [\mathbb{R}^7] \cong \mathfrak{g}_2 \oplus \mathbb{R}^7.$$

We define another map $\langle \cdot \rangle : \mathfrak{so}(7) \to \mathbb{R}^7$ by

$$\langle (a_{\ell m}) \rangle = (\epsilon_{\ell mn} a_{mn}).$$

This gives us a second characterization of $\mathfrak{g}_2$ since

$$\ker \langle \cdot \rangle = \mathfrak{g}_2$$

by Proposition 4.1. For the purpose of carrying out calculations, we note that for v and w in $\mathbb{R}^7$,

$$\begin{aligned} \langle [v] \rangle &= 6v \\ \langle [v][w] \rangle &= -3[v]w = 3[w]v. \end{aligned}$$

4.3 Recovering $\mathfrak{g}_2$ from its Dynkin Diagram

The Dynkin diagram of type G_2 is shown in Figure 4.1. From this diagram we generate the root system R for $\mathfrak{g}_2$. The root system as a subset of $\mathbb{R}^2$ corresponds to the roots of a adjoint representation of a Lie algebra. For details see any text on Lie Representations. We picture R in Figure 4.2. The values of the roots are given by

$$\alpha_1 = (1, 0), \alpha_2 = (-3/2, \sqrt{3}/2), \alpha_3 = (-1/2, \sqrt{3}/2),$$
$$\alpha_4 = (1/2, \sqrt{3}/2), \alpha_5 = (3/2, \sqrt{3}/2), \alpha_6 = (0, \sqrt{3}),$$

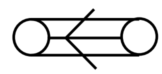

Figure 4.1: The Dynkin diagram for G_2.

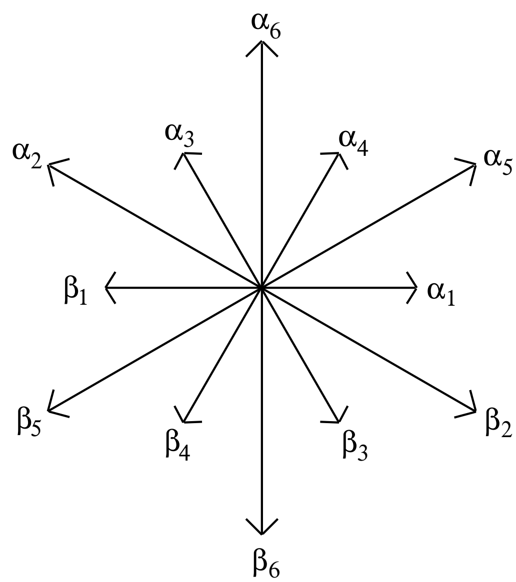

Figure 4.2: The root system associated to G_2.

and $\beta_i = -\alpha_i$ for $i = 1, \ldots, 6$. Following Fulton and Harris [11], we outline how to construct a basis for the 14-dimensional Lie algebra $\mathfrak{g}_2$. We begin by choosing X_1 and X_2 which are eigenvectors for the action of $\mathfrak{h}$ with eigenvalues α_1 and α_2 respectively. Similarly, take Y_1 and Y_2 with eigenvalues β_1 and β_2. The reason for the "X", "Y" terminology comes from the physics literature where they act as "raising" and "lowering" operators. Now define $H_1 = [X_1, Y_1]$ and $H_2 = [X_2, Y_2]$. It is possible to choose Y_1 and Y_2 so that $\alpha_1(H_1) = \alpha_2(H_2) = 2$ and thus

$$[H_1, X_1] = 2X_1 \quad \text{and} \quad [H_2, X_2] = 2X_2$$

so that

$$[H_1, Y_1] = -2Y_1 \quad \text{and} \quad [H_2, Y_2] = -2Y_2.$$

From a knowledge of the representation theory of $\mathfrak{sl}_2\mathbb{C}$, we see that $\{H_1, X_1, Y_1\}$ and $\{H_2, X_2, Y_2\}$ each span a subalgebra isomorphic to $\mathfrak{sl}_2\mathbb{C}$.

Now we see from the root diagram, Figure 4.2, that there are several relationships among the roots, such as $\alpha_3 = \alpha_1 + \alpha_2$. These relationships determine the relations $X_3 = [X_1, X_2]$, $X_4 = [X_1, X_3]$, $X_5 = [X_1, X_4]$, and $X_6 = [X_2, X_5]$. We have similar relations for the Y_i. The 14 elements H_1,H_2,X_1, $\cdots$, X_6, Y_1, $\cdots$, Y_6 form a basis for $\mathfrak{g}_2$.

All that remains now is to construct a multiplication table relating the 14 basis elements. This is no easy task, and we refer you to [11] for the details. We summarize the results in Figure 4.3.

This gives a complete accounting of a basis for $\mathfrak{g}_2$. The problem with this process is that choosing the operators H_1, X_1, Y_1 and H_2, X_2, Y_2 is not actually specified by Fulton and Harris. This turns out to be a non-trivial process so that generating an explicit basis for $\mathfrak{g}_2$ is difficult by their method. In the next section, we will see how to construct an explicit basis using a graph.

4.4 A Combinatorial Construction of $\mathfrak{g}_2$

In a recent work by N. J. Wildberger [20], a basis for $\mathfrak{g}_2$ is constructed by considering operators on the graph pictured in Figure 4.4, called the *G_2-hexagon*. The G_2-hexagon consists of vertices $v_\beta, v_{\beta\alpha\beta\beta}, v_{\beta\alpha\beta\beta\alpha\beta}, v_{\beta\alpha\beta\beta\alpha}, v_{\beta\alpha}, v_\phi$, and $v_{\beta\alpha\beta}$ which are connected by directed edges with weights $-2, -1, 1$, or 2, with 1 the default weight. Corresponding to the G_2-hexagon is Figure 4.5, a diagram similar to Figure 4.2 which defines vectors called *roots* for directions associated with edges in the G_2-hexagon. Figure 4.5 is in fact a root system for G_2, and it is this property that will allow us to construct a basis for $\mathfrak{g}_2$.

Constructing a basis for $\mathfrak{g}_2$ is a simple task with the G_2-hexagon. First we define an operator X_γ on the vertices where γ is a root from our root system. The rule is X_γ takes the vertex v to n times the vertex w whenever there is a an edge between v and w in the direction of γ with weight n. For example, $X_{\alpha+2\beta}$ sends v_ϕ to $2v_{\beta\alpha\beta\beta\alpha\beta}$ and $v_{\beta\alpha\beta\beta}$ to 0. We then have 12 unique operators X_γ; one for each root. We can define even more operators on the space of vertices using the bracket operation. In fact, we do just that, defining,

$$H_\gamma = [X_\gamma, X_{-\gamma}].$$

These operators lead us to the key theorem of Wildberger's paper.

	H_2	X_1	Y_1	X_2	Y_2	X_3	Y_3
H_1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3
H_2		$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$
X_1			H_1	X_3	0	X_4	$3Y_2$
Y_1				0	Y_3	$3X_2$	Y_4
X_2					H_2	0	$-Y_1$
Y_2						$-X_1$	0
X_3							$-H_1 - 3H_2$
Y_3							
X_4							
Y_4							
X_5							
Y_5							
X_6							

	X_4	Y_4	X_5	Y_5	X_6	Y_6
H_1	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H_2	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X_1	X_5	$4Y_3$	0	$3Y_4$	0	0
Y_1	$4X_3$	Y_5	$3X_4$	0	0	0
X_2	0	0	X_6	0	0	Y_5
Y_2	0	0	0	Y_6	X_5	0
X_3	$-X_6$	$4Y_1$	0	0	0	$3Y_4$
Y_3	$4X_1$	$-Y_6$	0	0	$3X_4$	0
X_4		$8H_1 + 12H_2$	0	$-12Y_1$	0	$12Y_3$
Y_4			$-12X_1$	0	$12X_3$	0
X_5				$-36H_1 - 36H_2$	0	$36Y_2$
Y_5					$36X_2$	0
X_6						$36H_1 + 72H_2$

Figure 4.3: A multiplication table for $\mathfrak{g}_2$ derived from the Dynkin diagram

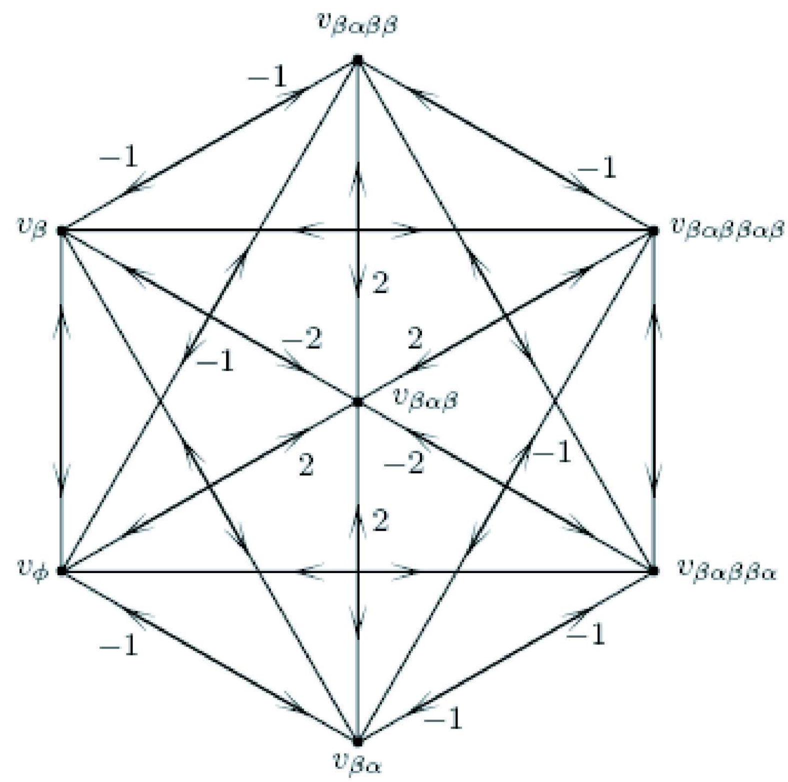

Figure 4.4: The G_2-hexagon

Theorem 4.1. *The span of the operators* $\{X_\gamma, H_\gamma \mid \gamma \text{ is a root}\}$ *is closed under brackets and forms a 14-dimensional Lie algebra isomorphic to* $\mathfrak{g}_2$*. In particular, a basis of* $\mathfrak{g}_2$ *is* $\{X_\gamma \mid \gamma \text{ is a root}\} \cup \{H_\alpha, H_\beta\}$.

Constructing the given operators on the vertices of the G_2-hexagon is equivalent to constructing $\mathfrak{g}_2$. This process is particularly attractive because finding explicit matrices that form a basis for $\mathfrak{g}_2$ amounts to finding the matrix representations of the operators with respect to the vertices of the G_2-hexagon.

We now present an explicit basis for $\mathfrak{g}_2$. We again use the physics notation of splitting the twelve operators X_γ into six operators X_γ and six

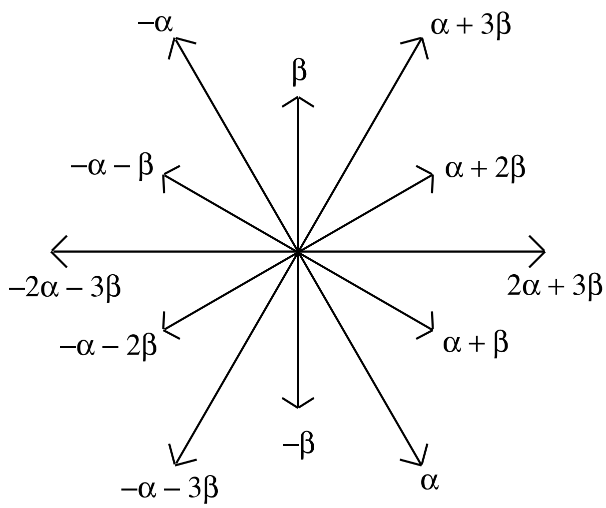

Figure 4.5: Another root system associated to G_2.

operators Y_γ where γ is always positive.[1] The fourteen basis elements are

$$X_\alpha = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0 \end{pmatrix}, \; Y_\alpha = \begin{pmatrix} 0&0&0&0&0&0&2\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},$$

$$X_\beta = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \; Y_\beta = \begin{pmatrix} 0&0&0&2&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0 \end{pmatrix},$$

[1]Here $Y_\gamma = X_{-\gamma}$ where X_γ is from of our original set of twelve operators.

$$X_{\alpha+\beta} = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \; Y_{\alpha+\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_{2\alpha+3\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; Y_{2\alpha+3\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_{\alpha+2\beta} = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; Y_{\alpha+2\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$X_{\alpha+3\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; Y_{\alpha+3\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$H_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; H_{\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

In general, an element of $\mathfrak{g}_2$ may be written in the form

$$\begin{pmatrix} 0 & -2\,a_5 & 2\,a_9 & 2\,a_4 & 2\,a_{10} & -2\,a_6 & 2\,a_2 \\ a_6 & -a_{13}+a_{14} & a_2 & -a_{10} & a_8 & 0 & a_3 \\ a_{10} & a_4 & -a_{14} & -a_{12} & 0 & a_8 & -a_6 \\ a_2 & -a_9 & a_{11} & -a_{13}+2\,a_{14} & a_6 & a_3 & 0 \\ a_9 & a_7 & 0 & -a_5 & a_{14} & a_2 & -a_{11} \\ a_5 & 0 & a_7 & a_1 & a_4 & a_{13}-a_{14} & -a_9 \\ a_4 & a_1 & a_5 & 0 & a12 & -a_{10} & a_{13}-2\,a_{14} \end{pmatrix}$$

for constants a_i.

We notice a problem with these matrices. Recall that the Lie algebra $\mathfrak{so}(7)$ is the space of all 7×7 skew-symmetric matrices. We know from previous considerations that $\mathfrak{g}_2 \subset \mathfrak{so}(7)$. Yet none of the basis elements we have constructed are skew-symmetric! The reason for this is that we have constructed a space isomorphic to $\mathfrak{g}_2$, and not $\mathfrak{g}_2$ as it was defined. The solution to the problem is a change of basis. We leave it for future work to find such a change of basis since there is a more immediate geometric construction of a $\mathfrak{g}_2$ basis given in Chapter 5.1.

4.5 Summary

In Figure 4.6 we summarize the different ways we can view $\mathfrak{g}_2$. For information on $\mathfrak{g}_2$ as the kernel of a derivation see [14; 3].

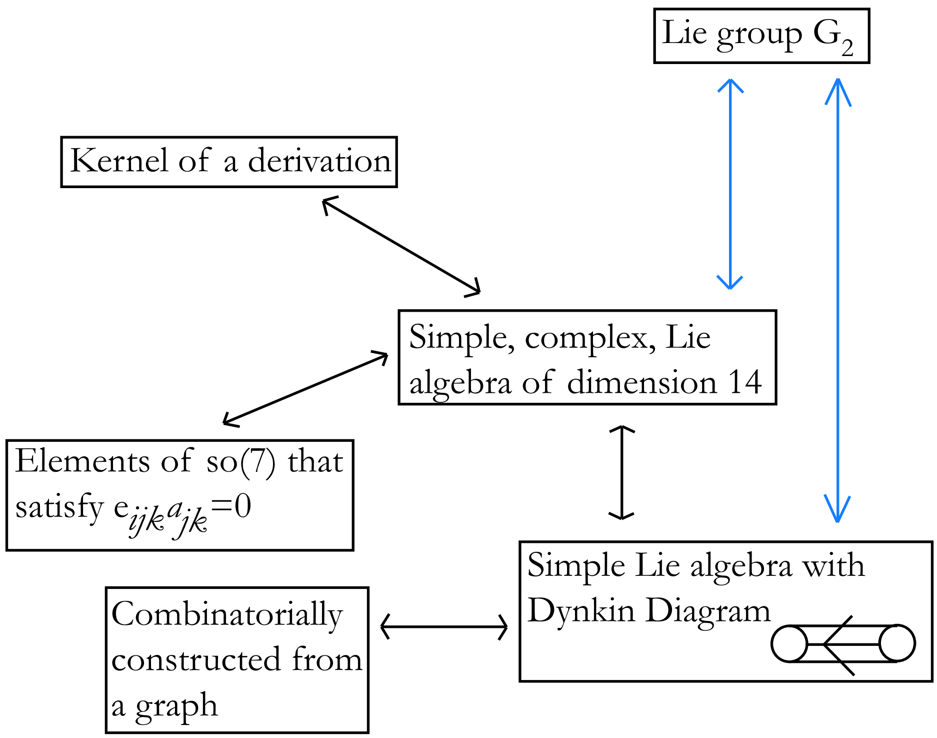

Figure 4.6: Relationships between the ways of viewing $\mathfrak{g}_2$

Chapter 5

Results: Relating G_2 and $\mathfrak{g}_2$

5.1 Generating $\mathfrak{g}_2$ using the $\epsilon_{\ell mn}$

In Chapter 4 we gave a description of $\mathfrak{g}_2$ using a set of constants labeled $\epsilon_{\ell mn}$. A quick glance at this result, Proposition 4.1, doesn't immediately suggest a way to generate elements of $\mathfrak{g}_2$. Yet if we rephrase the result of the proposition in the language of matrices it seems reasonable that we might learn something new. Recall that $M(n, n)$, the set of $n \times n$ matrices, can be made into an inner product space using the inner product

$$\langle A, B \rangle = a_{\ell m} b_{\ell m}$$

for $A = (a_{\ell m}), B = (b_{\ell m}) \in M(n, n)$.

Let $E_\ell = (\epsilon_{\ell mn})$. The explicit matrices are

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, E_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{and } E_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and we note that all these matrices lie in $\mathfrak{so}(7) \subset M(7,7)$ and are mutually orthogonal. Take $A = (a_{\ell m})$ to be in $\mathfrak{so}(7)$. Then we note the equivalence,

$$\epsilon_{\ell mn} a_{mn} = 0 \longleftrightarrow \langle A, E_\ell \rangle = 0,$$

for all $\ell = 1, \cdots, 7$. Proposition 4.1 is really just a geometric statement. It says that a matrix lies in $\mathfrak{g}_2$ if and only if that matrix is orthogonal to the space spanned by $E_1, \cdots, E_7$.

We recall that every matrix can be decomposed into a sum of a symmetric matrix and a skew symmetric matrix. That is,

$$M(7,7) = \mathrm{Sym}(7,7) \oplus \mathrm{Skew}(7,7) \cong \mathrm{Sym}(7,7) \oplus \mathfrak{so}(7).$$

Now $\dim M(n,n) = n^2$ and $\dim \mathrm{Sym}(n,n) = n(n+1)/2$ so that $\dim \mathfrak{so}(n) = n(n-1)/2$. Further, since matrices in $\mathfrak{g}_2$ are orthogonal to $E_1, \ldots, E_7$ in $\mathfrak{so}(7)$, we see

$$\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathrm{span}\,\{E_1, \cdots, E_7\}.$$

This implies $\dim \mathfrak{g}_2 = 7 \times 6/2 - 7 = 14$, as expected.

Now constructing a basis for $\mathfrak{g}_2$ amounts to finding 14 linearly independent matrices A that satisfy $\langle A, E_\ell \rangle = 0$ for $\ell = 1, \ldots, 7$. We present one quick way of choosing these A. We note that each of the E_ℓ has 6 nonzero entries, and the value at the position of these entries is zero in the other E_ℓ. Now for each E_ℓ, take two pairs of skew symmetric entries (entries of the form $a_{mn} = -a_{nm}$) and construct a new matrix X_ℓ with these by swapping

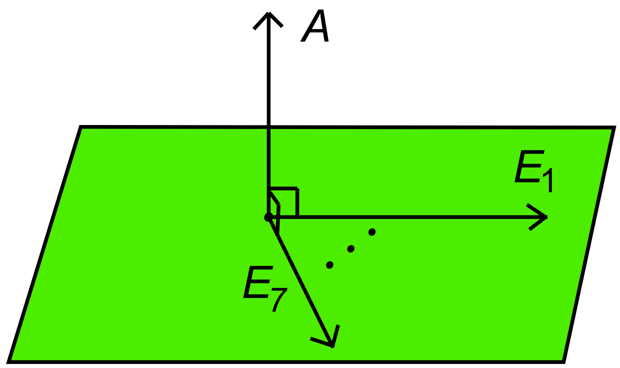

Figure 5.1: Every matrix A in $\mathfrak{g}_2$ is orthogonal to the span of $\{E_1, \ldots, E_7\}$

the signs on a pair. For example, from E_1 we can take the entries $(2,3)$ and $(3,2)$ and the opposite signed $(4,5)$ and $(5,4)$ to form

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \mapsto X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now take the unused skew symmetric pair, and one of the two used pairs. Swap the sign on the entries of the unused pair to make Y_ℓ. Continuing with E_1, we take the opposite signed $(7,6)$ and $(6,7)$, and $(4,5)$ and $(5,4)$ to form

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \mapsto Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

We notice that X_1 and Y_1 by construction are orthogonal to E_1. Examining $x_1X_1 + y_1Y_1 = 0$ shows that x_1 and y_1 are forced to zero so that X_1 and Y_1 are linearly independent. Further, because no nonzero entry of X_1 or Y_1

corresponds to a nonzero entry of $E_2, \ldots,$ or E_7, the sets $\{X_1, E_2, \ldots, E_7\}$ and $\{Y_1, E_2, \ldots, E_7\}$ are each mutually orthogonal. Thus X_1 and Y_1 are linearly independent elements of $\mathfrak{g}_2$. Using the described algorithm, the matrices $X_2, \ldots, X_7, Y_1, \ldots, Y_7$ are as follows,

$$X_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_5 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Y_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } Y_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that each pair (X_ℓ, Y_ℓ) are linearly independent elements of $\mathfrak{g}_2$, and moreover, $\{X_1, \ldots, X_7, Y_1, \ldots, Y_7\}$ forms a linearly independent set. This argument proves the following key theorem.

Theorem 5.1. *The set of matrices* $\mathcal{S} = \{X_1, \ldots, X_7, Y_1, \ldots, Y_7\}$ *form a basis for* $\mathfrak{g}_2$.

Note that the basis is not orthogonal since $[X_i, Y_i] = \pm 2$ for $i = 1, \ldots, 7$. A general element of $\mathfrak{g}_2$ can be written in the form

$$\begin{pmatrix} 0 & x_3 & -x_2 & x_5 & -x_4 & -x_7 & -x_6 + y_6 \\ -x_3 & 0 & x_1 & x_6 & -x_7 + y_7 & x_4 - y_4 & x_5 + y_5 \\ x_2 & -x_1 & 0 & -y_7 & y_6 & y_5 & y_4 \\ -x_5 & -x_6 & y_7 & 0 & -x_1 + y_1 & -x_2 + y_2 & -x_3 + y_3 \\ x_4 & x_7 - y_7 & -y_6 & x_1 - y_1 & 0 & y_3 & -y_2 \\ x_7 & -x_4 + y_4 & -y_5 & x_2 - y_2 & -y_3 & 0 & y_1 \\ x_6 - y_6 & -x_5 - y_5 & -y_4 & x_3 - y_3 & y_2 & -y_1 & 0 \end{pmatrix}.$$

We will examine the properties of this basis and use it construct portions of G_2 in the sections that follow.

5.2 Commutativity of Basis Elements

In this section we examine commutation relations for the elements of $\mathcal{S}$. The commutation table is shown in Figure 5.2. Two properties immediately stand out in contrast to the commutation table constructed using the

	X_2	X_3	X_4	X_5	X_6	X_7
X_1	$-X_2+Y_3$	X_2+Y_2	X_5-Y_5	$-X_4-Y_4$	$-X_7+Y_7$	X_6+Y_6
X_2		$-X_1-Y_1$	$-X_6-Y_6$	$-X_7-Y_7$	$-Y_4$	X_5-Y_5
X_3			$-X_7$	$-2X_6$	$2X_5$	X_4
X_4				$-Y_1$	Y_2	$-2X_3-2Y_3$
X_5					$-X_3$	Y_2
X_6						Y_1
X_7						
Y_1						
Y_2						
Y_3						
Y_4						
Y_5						
Y_6						

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7
X_1	0	Y_3	$-Y_2$	Y_5	$-Y_4$	Y_7	$-2X_6-2Y_6$
X_2	$-Y_3$	0	Y_1	X_6	X_7+Y_7	X_4+Y_4	X_5-Y_5
X_3	Y_2	$-Y_1$	0	X_7+Y_7	$-X_6$	$-X_5$	$-X_4-Y_4$
X_4	X_5	$-X_6+Y_6$	$-X_7$	0	$-X_1-Y_1$	$-X_2-Y_2$	X_3+Y_3
X_5	$-X_4$	$-X_7$	$-X_6+X_6$	$-X_1-Y_1$	0	$-X_3$	$-X_2-Y_2$
X_6	$-X_7$	X_4	X_5	$-X_2$	X_3	0	X_1
X_7	X_6	X_5	$-X_4$	$-X_3-Y_3$	$-X_2-Y_2$	X_1+Y_1	0
Y_1		$-2Y_3$	$2Y_2$	Y_5	$-Y_4$	$-X_7-Y_7$	X_6+Y_6
Y_2			$-2Y_1$	$-X_6-Y_6$	$-Y_7$	X_4+Y_4	Y_5
Y_3				$-Y_7$	X_6+Y_6	X_5-Y_5	Y_4
Y_4					$2X_1+2Y_1$	$-X_2-Y_2$	$-Y_3$
Y_5						X_3+Y_3	$-Y_2$
Y_6							X_1

Figure 5.2: Commutation table for $\mathfrak{g}_2$ using $\mathbb{S}$.

Dynkin diagram (Figure 4.3). First, the only non-trivial commutating pairs are X_i and Y_i for $i = 1, \ldots, 7$. Second, $0, \pm 1, \pm 2$ are the only coefficients seen in the table. Therefore the disadvantage of $\mathbb{S}$ compared to the Dynkin basis is that fewer elements commute, and thus taking the exponential of sums of pairs will be more complicated. The advantage of the basis $\mathbb{S}$ is that simple coefficients in the table should lead to simpler results when we eventually exponential down to G_2.

5.3 Constructing a Matrix Representation of G_2

We can examine the structure of G_2 by considering the image of subsets of $\mathbb{S}$ under the exponential map. In particular, we give the results for individual elements of $\mathbb{S}$ as well as commuting pairs in Appendix A. Note that $SO(2) \cong S^1$ so that

$$S^1 = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}.$$

Thus the 2-torus T^2 can be expressed as the direct product

$$S^1 \times S^1 = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\varphi & \sin\varphi \\ 0 & 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \right\}.$$

This is relevant because we see these structures in the matrices of Appendix A. Specifically, $\exp(x_i X_i)$ and $\exp(y_i Y_i)$ have the structure of $S^1 \times S^1 \cong T^2$ and $\exp(x_i X_i + y_i Y_i)$ look like $S^1 \times S^1 \times S^1 \cong T^3$. As Curtis [9] points out, the matrices $\exp(x_i X_i + y_i Y_i)$ are maximal tori in $SO(7)$, and since $G_2 \subset SO(7)$, $\exp(x_i X_i + y_i Y_i)$ are also maximal tori in G_2. As Curtis shows, since G_2 is connected it can be covered by maximal tori. That is,

$$G_2 = \bigcup_{g \in G} g \exp(x_i X_i + y_i Y_i) g^{-1} \text{ for } i = 1, \cdots, 7.$$

Therefore we have yet another characterization of G_2.

Now that we have a complete description of $\mathfrak{g}_2$ from the previous section, we can use the exponential map to generate G_2. Since G_2 is connected, the most obvious way to do this is to use the canonical coordinates of the first kind,

$$G_2 = \left\{ \exp\left(\sum_{i=1}^{7} x_i X_i + \sum_{i=1}^{7} y_i Y_i \right) \mid x_i, y_i \in \mathbb{R} \right\}.$$

This is an accurate accounting of G_2, yet presents many problems in terms of computing explicit elements of G_2. In practical terms, it is a calculation that the current version of Maple can not do.

Thankfully, canonical coordinates of the second kind are easily computed and also generate all of G_2. That is,

$$G_2 = \left\{ \left(\prod_{n=1}^{7} \exp(x_i X_i) \right) \cdot \left(\prod_{n=1}^{7} \exp(y_i Y_i) \right) \mid x_i, y_i \in \mathbb{R} \right\}.$$

We compile a 14-parameter characterization of G_2 using this method in Appendix B.

5.4 Examples

This section is comprised of examples utilizing the basis $\mathbb{S}$. For example, we show how certain elements of G_2 constructed using $\mathbb{S}$ satisfy the conditions imposed in Chapter 3. Recall that Appendix B contains the image of elements of $\mathbb{S}$ under the exponential map.

Example 1. Recall from Section 3.2.2 that G_2 is the automorphism group of the octonions. Thus, for all $g \in \exp(\mathbb{S})$ we must have

$$g(ab) = g(a)g(b) \quad \text{for all } a, b \in \mathbb{O}.$$

Now g is a 7-by-7 matrix, yet $\dim \mathbb{O} = 8$. We can avoid this problem by noting that the real part of $\mathbb{O}$ does not matter since real numbers are invariant under such g. Thus we need only consider a and b in $\operatorname{Im} \mathbb{O} \cong \mathbb{R}^7$. Consider $a = e_1 = i$, $b = e_2 = j$, and $g = \exp(x_1 X_1)$. Then we calculate

$$\begin{aligned}
g(ij) &= g(k) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos x_1 & \sin x_1 & 0 & 0 & 0 & 0 \\ 0 & -\sin x_1 & \cos x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos x_1 & -\sin x_1 & 0 & 0 \\ 0 & 0 & 0 & \sin x_1 & \cos x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sin x_1 & \cos x_1 & 0 & 0 & 0 & 0 \end{pmatrix}^T \\
&= \sin(x_1) j + \cos(x_1) k
\end{aligned}$$

but

$$\begin{aligned} g(i) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos x_1 & \sin x_1 & 0 & 0 & 0 & 0 \\ 0 & -\sin x_1 & \cos x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos x_1 & -\sin x_1 & 0 & 0 \\ 0 & 0 & 0 & \sin x_1 & \cos x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \\ &= i \end{aligned}$$

and

$$\begin{aligned} g(j) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos x_1 & \sin x_1 & 0 & 0 & 0 & 0 \\ 0 & -\sin x_1 & \cos x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos x_1 & -\sin x_1 & 0 & 0 \\ 0 & 0 & 0 & \sin x_1 & \cos x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \cos x_1 & -\sin x_1 & 0 & 0 & 0 & 0 \end{pmatrix}^T \\ &= \cos(x_1)j - sin(x_1)k. \end{aligned}$$

Thus we have

$$g(i)g(j) = i[\cos(x_1)j - \sin(x_1)k] = \sin(x_1)j + \cos(x_1)k = g(ij)$$

as desired.

Example 2. Our next example involves the cross product characterization of G_2 described in Section 3.3. In this case, we should have $g(a \times b) = g(a) \times g(b)$ for g in $\exp(\mathbb{S})$. Consider $g = \exp(y_3 Y_3)$ in G_2 and take $a = e_4$

and $b = e_7$. Then

$$\begin{aligned}
g(e_4 \times e_7) &= g(e_3) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos y_3 & 0 & 0 & \sin y_3 \\ 0 & 0 & 0 & 0 & \cos y_3 & \sin y_3 & 0 \\ 0 & 0 & 0 & 0 & -\sin y_3 & \cos y_3 & 0 \\ 0 & 0 & 0 & -\sin y_3 & 0 & 0 & \cos y_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^T \\
&= e_3
\end{aligned}$$

but also

$$\begin{aligned}
g &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos y_3 & 0 & 0 & \sin y_3 \\ 0 & 0 & 0 & 0 & \cos y_3 & \sin y_3 & 0 \\ 0 & 0 & 0 & 0 & -\sin y_3 & \cos y_3 & 0 \\ 0 & 0 & 0 & -\sin y_3 & 0 & 0 & \cos y_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & \cos y_3 & 0 & 0 & -\sin y_3 \end{pmatrix}^T \\
&= \cos(y_3)e_4 - \sin(y_3)e_7
\end{aligned}$$

and

$$\begin{aligned}
g(e_7) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos y_3 & 0 & 0 & \sin y_3 \\ 0 & 0 & 0 & 0 & \cos y_3 & \sin y_3 & 0 \\ 0 & 0 & 0 & 0 & -\sin y_3 & \cos y_3 & 0 \\ 0 & 0 & 0 & -\sin y_3 & 0 & 0 & \cos y_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & \sin y_3 & 0 & 0 & \cos y_3 \end{pmatrix}^T \\
&= \sin(y_3)e_4 + \cos(y_3)e_7.
\end{aligned}$$

Therefore

$$\begin{aligned} g(e_4)\times g(e_7) &= (\cos(y_3)e_4 - \sin(y_3)e_7)\times(\sin(y_3)e_4 + \cos(y_3)e_7) \\ &= -\sin^2(y_3)(e_7\times e_4) + \cos^2(y_3)(e_4\times e_7) \\ &= [\sin^2(y_3) + \cos^2(y_3)](e_4\times e_7) \\ &= e_4\times e_7 = e_3 \end{aligned}$$

so that

$$g(e_4\times e_7) = e_3 = g(e_4)\times g(e_7).$$

That is, g preserves the cross product.

Example 3. As a final example, we check that the associative calibration from Section 3.4 is preserved by selected elements of $\exp\mathbb{S}$. Recall that the associative calibration is defined as $\phi(a,b,c) = \langle a, bc\rangle$ for a, b, and c in $\operatorname{Im}\mathbb{O}$. Take $a = e_1$, $b = e_6$, and $c = e_7$ and consider the action of $g = \exp(y_7Y_7)$ on ϕ. Then

$$\begin{aligned} \{g^*\phi\}(e_1,e_2,e_7) &= \phi(g(e_1), g(e_6), g(e_7)) \\ &= \langle\cos(y_7)e_1 + \sin(y_7)e_6, [-\sin(y_7)e_1 + \cos(y_7)e_6]e_7\rangle \\ &= \langle\cos(y_7)e_1 + \sin(y_7)e_6, -\sin(y_7)e_6 + \cos(y_7)e_1\rangle \\ &= -\sin^2 y_7 - cos^2 y_7 = -1. \end{aligned}$$

But

$$\begin{aligned} \phi(e_1,e_6,e_7) &= \langle e_1, e_6e_7\rangle \\ &= -\langle e_1, e_1\rangle = -1 \end{aligned}$$

so that

$$g^*\phi\}(e_1,e_2,e_7) = \phi(e_1,e_6,e_7)$$

as we would hope.

5.5 An Area for Future Research: Spin(7)

Our geometric characterization of $\mathfrak{g}_2$ given in Section 5.1 is a novel way of considering a Lie algebra. Previously, the Dynkin diagram method of constructing a Lie algebra was the only major way to construct a representation. It stands to reason that examining where our method comes from will lead to explicit characterizations of other Lie algebras and their corresponding Lie groups.

Our geometric method relied on Robert Bryant's $\epsilon_{\ell mn}$ characterization of $\mathfrak{g}_2$ described in Section 4.2. These $\epsilon_{\ell mn}$ in turn come from the associative calibration ϕ. We recall that g is in G_2 if and only if $g^*\phi = \phi$. Thus there should be some connection between calibrations and geometric descriptions of Lie algebras.

This immediately suggest a new area of research. Recall the definition of the coassociative calibration given in Section 3.4.3. It is well known that the group Spin(7) preserves the coassociative calibration [4]. We suspect that some characterization of Spin(7)'s Lie algebra can be given using the coefficients of the coassociative calibration. From there, a geometric characterization of Spin(7)'s Lie algebra can't be far.

Appendix A

Exponential of Elements and Commuting Pairs of $\mathbb{S}$

This appendix catalogs the elements of G_2 that are images of elements of the basis $\mathbb{S}$ of $\mathfrak{g}_2$.

$$\exp(x_1X_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos x_1 & \sin x_1 & 0 & 0 & 0 & 0 \\ 0 & -\sin x_1 & \cos x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos x_1 & -\sin x_1 & 0 & 0 \\ 0 & 0 & 0 & \sin x_1 & \cos x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\exp(x_2X_2) = \begin{pmatrix} \cos x_2 & 0 & -\sin x_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sin x_2 & 0 & \cos x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos x_2 & 0 & -\sin x_2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sin x_2 & 0 & \cos x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\exp(x_3X_3) = \begin{pmatrix} \cos x_3 & \sin x_3 & 0 & 0 & 0 & 0 & 0 \\ -\sin x_3 & \cos x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos x_3 & 0 & 0 & -\sin x_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin x_3 & 0 & 0 & \cos x_3 \end{pmatrix}$$

$$\exp(x_4X_4) = \begin{pmatrix} \cos x_4 & 0 & 0 & 0 & -\sin x_4 & 0 & 0 \\ 0 & \cos x_4 & 0 & 0 & 0 & \sin x_4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \sin x_4 & 0 & 0 & 0 & \cos x_4 & 0 & 0 \\ 0 & -\sin x_4 & 0 & 0 & 0 & \cos x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\exp(x_5X_5) = \begin{pmatrix} \cos x_5 & 0 & 0 & \sin x_5 & 0 & 0 & 0 \\ 0 & \cos x_5 & 0 & 0 & 0 & 0 & \sin x_5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\sin x_5 & 0 & 0 & \cos x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\sin x_5 & 0 & 0 & 0 & 0 & \cos x_5 \end{pmatrix}$$

$$\exp(x_6X_6) = \begin{pmatrix} \cos x_6 & 0 & 0 & 0 & 0 & 0 & -\sin x_6 \\ 0 & \cos x_6 & 0 & \sin x_6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\sin x_6 & 0 & \cos x_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \sin x_6 & 0 & 0 & 0 & 0 & 0 & \cos x_6 \end{pmatrix}$$

$$\exp(x_7X_7) = \begin{pmatrix} \cos x_7 & 0 & 0 & 0 & 0 & -\sin x_7 & 0 \\ 0 & \cos x_7 & 0 & 0 & -\sin x_7 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sin x_7 & 0 & 0 & \cos x_7 & 0 & 0 \\ \sin x_7 & 0 & 0 & 0 & 0 & \cos x_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\exp(y_1Y_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos y_1 & \sin y_1 & 0 & 0 \\ 0 & 0 & 0 & -\sin y_1 & \cos y_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos y_1 & \sin y_1 \\ 0 & 0 & 0 & 0 & 0 & -\sin y_1 & \cos y_1 \end{pmatrix}$$

$$\exp(y_2 Y_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos y_2 & 0 & \sin y_2 & 0 \\ 0 & 0 & 0 & 0 & \cos y_2 & 0 & -\sin y_2 \\ 0 & 0 & 0 & -\sin y_2 & 0 & \cos y_2 & 0 \\ 0 & 0 & 0 & 0 & \sin y_2 & 0 & \cos y_2 \end{pmatrix}$$

$$\exp(y_3 Y_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos y_3 & 0 & 0 & \sin y_3 \\ 0 & 0 & 0 & 0 & \cos y_3 & \sin y_3 & 0 \\ 0 & 0 & 0 & 0 & -\sin y_3 & \cos y_3 & 0 \\ 0 & 0 & 0 & -\sin y_3 & 0 & 0 & \cos y_3 \end{pmatrix}$$

$$\exp(y_4 Y_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos y_4 & 0 & 0 & 0 & -\sin y_4 & 0 \\ 0 & 0 & \cos y_4 & 0 & 0 & 0 & \sin y_4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sin y_4 & 0 & 0 & 0 & \cos y_4 & 0 \\ 0 & 0 & -\sin y_4 & 0 & 0 & 0 & \cos y_4 \end{pmatrix}$$

$$\exp(y_5 Y_5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos y_5 & 0 & 0 & 0 & 0 & \sin y_5 \\ 0 & 0 & \cos y_5 & 0 & 0 & \sin y_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\sin y_5 & 0 & 0 & \cos y_5 & 0 \\ 0 & -\sin y_5 & 0 & 0 & 0 & 0 & \cos y_5 \end{pmatrix}$$

$$\exp(y_6 Y_6) = \begin{pmatrix} \cos y_6 & 0 & 0 & 0 & 0 & 0 & \sin y_6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos y_6 & 0 & \sin y_6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\sin y_6 & 0 & \cos y_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\sin y_6 & 0 & 0 & 0 & 0 & 0 & \cos y_6 \end{pmatrix}$$

$$\exp(y_7Y_7) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos y_7 & 0 & 0 & \sin y_7 & 0 & 0 \\ 0 & 0 & \cos y_7 & -\sin y_7 & 0 & 0 & 0 \\ 0 & 0 & \sin y_7 & \cos y_7 & 0 & 0 & 0 \\ 0 & -\sin y_7 & 0 & 0 & \cos y_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\exp(x_1X_1 + y_1Y_1) =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos x_1 & \sin x_1 & 0 & 0 & 0 & 0 \\ 0 & -\sin x_1 & \cos x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(x_1 - y_1) & -\sin(x_1 - y_1) & 0 & 0 \\ 0 & 0 & 0 & \sin(x_1 - y_1) & \cos(x_1 - y_1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos y_1 & \sin y_1 \\ 0 & 0 & 0 & 0 & 0 & -\sin y_1 & \cos y_1 \end{pmatrix}$$

$$\exp(x_2X_2 + y_2Y_2) =$$

$$\begin{pmatrix} \cos x_2 & 0 & -\sin x_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sin x_2 & 0 & \cos x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(-x_2 + y_2) & 0 & \sin(-x_2 + y_2) & 0 \\ 0 & 0 & 0 & 0 & \cos y_2 & 0 & -\sin y_2 \\ 0 & 0 & 0 & -\sin(-x_2 + y_2) & 0 & \cos(-x_2 + y_2) & 0 \\ 0 & 0 & 0 & 0 & \sin y_2 & 0 & \cos y_2 \end{pmatrix}$$

$$\exp(x_3X_3 + y_3Y_3) =$$

$$\begin{pmatrix} \cos x_3 & \sin x_3 & 0 & 0 & 0 & 0 & 0 \\ -\sin x_3 & \cos x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(-x_3 + y_3) & 0 & 0 & \sin(-x_3 + y_3) \\ 0 & 0 & 0 & 0 & \cos y_3 & \sin y_3 & 0 \\ 0 & 0 & 0 & 0 & -\sin y_3 & \cos y_3 & 0 \\ 0 & 0 & 0 & -\sin(-x_3 + y_3) & 0 & 0 & \cos(-x_3 + y_3) \end{pmatrix}$$

$\exp(x_4X_4 + y_4Y_4) =$

$$\begin{pmatrix} \cos x_4 & 0 & 0 & 0 & -\sin x_4 & 0 & 0 \\ 0 & \cos(-x_4+y_4) & 0 & 0 & 0 & -\sin(-x_4+y_4) & 0 \\ 0 & 0 & \cos y_4 & 0 & 0 & 0 & \sin y_4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \sin x_4 & 0 & 0 & 0 & \cos x_4 & 0 & 0 \\ 0 & \sin(-x_4+y_4) & 0 & 0 & 0 & \cos(-x_4+y_4) & 0 \\ 0 & 0 & -\sin y_4 & 0 & 0 & 0 & \cos y_4 \end{pmatrix}$$

$\exp(x_5X_5 + y_5Y_5) =$

$$\begin{pmatrix} \cos x_5 & 0 & 0 & \sin x_5 & 0 & 0 & 0 \\ 0 & \cos(x_5+y_5) & 0 & 0 & 0 & 0 & \sin(x_5+y_5) \\ 0 & 0 & \cos y_5 & 0 & 0 & \sin y_5 & 0 \\ -\sin x_5 & 0 & 0 & \cos x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\sin y_5 & 0 & 0 & \cos y_5 & 0 \\ 0 & -\sin(x_5+y_5) & 0 & 0 & 0 & 0 & \cos(x_5+y_5) \end{pmatrix}$$

$\exp(x_6X_6 + y_6Y_6) =$

$$\begin{pmatrix} \cos(x_6-y_6) & 0 & 0 & 0 & 0 & 0 & -\sin(x_6-y_6) \\ 0 & \cos x_6 & 0 & \sin x_6 & 0 & 0 & 0 \\ 0 & 0 & \cos y_6 & 0 & \sin y_6 & 0 & 0 \\ 0 & -\sin x_6 & 0 & \cos x_6 & 0 & 0 & 0 \\ 0 & 0 & -\sin y_6 & 0 & \cos y_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \sin(x_6-y_6) & 0 & 0 & 0 & 0 & 0 & \cos(x_6-y_6) \end{pmatrix}$$

$\exp(x_7X_7 + y_7Y_7) =$

$$\begin{pmatrix} \cos x_7 & 0 & 0 & 0 & 0 & -\sin x_7 & 0 \\ 0 & \cos(x_7-y_7) & 0 & 0 & -\sin(x_7-y_7) & 0 & 0 \\ 0 & 0 & \cos y_7 & -\sin y_7 & 0 & 0 & 0 \\ 0 & 0 & \sin y_7 & \cos y_7 & 0 & 0 & 0 \\ 0 & \sin(x_7-y_7) & 0 & 0 & \cos(x_7-y_7) & 0 & 0 \\ \sin x_7 & 0 & 0 & 0 & 0 & \cos x_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Appendix B

A Matrix Representation of G_2

What follows is the canonical coordinates of the second kind of G_2 using the basis S from Theorem 5.1. This is analogous to the 1-parameter matrix description of SO(2),

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

We parameterize G_2 using the 14 parameters $x_1, \ldots, x_7, y_1, \ldots, y_7$. $G_2[i, j]$ is the i, j-th entry in the matrix representation. We present only the first two rows as an example since these entries can be easily computed with the equations given.

$G_2[1, 1] =$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+ \sin y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$+ \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \cos y_6 \cos x_2 \sin x_3 \sin x_4 \sin x_7$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$+ \sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$

$- \sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \sin y_6 \cos y_5 \sin x_2 \sin y_4$
$+ \sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+ \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \cos y_6 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \cos y_6 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_6 \sin y_5 \cos y_4 \cos x_2 \cos x_3 \sin x_4 \sin x_7$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$

$G_2[1, 2] =$
$\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$

$$-\cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+\sin y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_2 \cos x_3 \sin x_4 \sin x_7$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\sin y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$-\sin y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+\sin y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\sin y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$+\sin y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+\sin y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$+\sin y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$-\sin y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$+\sin y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$-\sin y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+\sin y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+\sin y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+\sin y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\sin y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+\sin y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$-\sin y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\sin y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+\sin y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$+\sin y_7 \cos y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin x_2 \cos y_4$$
$$-\sin y_7 \cos y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$-\sin y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$

$$
\begin{aligned}
&+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6\\
&+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7\\
&+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6\\
&+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6\\
&-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6\\
&-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6\\
&-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6\\
&+\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7\\
&-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6\\
&+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7\\
&+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6\\
&-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6\\
&+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6\\
&-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7\\
&+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6\\
&+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7\\
&+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6\\
&+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6\\
&-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6\\
&+\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7\\
&+\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6\\
&+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6\\
&-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7\\
&+\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6\\
&-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6\\
&+\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6\\
&-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6\\
&-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7\\
&+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6\\
&-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6\\
&+\cos y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6
\end{aligned}
$$

$$- \cos y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$- \cos y_7 \cos y_5 \cos y_4 \cos x_2 \cos x_3 \sin x_4 \sin x_7$$
$$+ \cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+ \cos y_7 \sin y_5 \sin x_2 \sin y_4$$
$$+ \cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$- \cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+ \cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$- \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$- \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$- \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$- \cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$- \cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$

$G_2[1,3] =$

$$\cos y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$+ \cos y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$- \cos y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$$
$$- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$- \cos y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$$
$$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$- \cos y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$

$$
\begin{aligned}
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \sin y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \sin y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \sin y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \sin y_7 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&- \sin y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \sin y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \sin y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \sin y_7 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&- \sin y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \sin y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \sin y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6
\end{aligned}
$$

$+\sin y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$-\sin y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+\sin y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$-\sin y_7 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$+\sin y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+\sin y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$-\sin y_7 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+\sin y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$-\sin y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$+\cos y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+\cos y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$-\cos y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+\cos y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$-\cos y_7 \cos y_6 \cos y_5 \sin x_2 \cos y_4$
$-\cos y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$-\cos y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$-\cos y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+\cos y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$-\cos y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_2 \cos x_3 \sin x_4 \sin x_7$
$-\cos y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+\cos y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$+\cos y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+\cos y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+\cos y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$+\cos y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$-\cos y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$-\cos y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+\cos y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$

$G_2[1,4] =$
$\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$-\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$-\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$-\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$-\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$

$$
\begin{aligned}
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \sin y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \sin y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \sin y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&- \sin y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \sin y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \sin y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \sin y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \sin y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&- \sin y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \sin y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \sin y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_7 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&- \cos y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \sin y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \sin y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \sin y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \sin y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \sin y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \sin y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \sin y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \sin y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_2 \cos x_3 \sin x_4 \sin x_7 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&+ \sin y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \sin y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7
\end{aligned}
$$

$- \sin y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \sin y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+ \cos y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \cos y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$- \cos y_7 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$- \cos y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \cos y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \cos y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \cos y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \cos y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \cos y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$- \cos y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \cos y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$- \cos y_7 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \cos y_7 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+ \cos y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \cos y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$+ \sin y_7 \cos y_6 \cos y_5 \sin x_2 \cos y_4$

$G_2[1,5] =$
$\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$

$$
\begin{aligned}
&- \sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&+ \sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&- \sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin x_2 \cos y_4 \\
&+ \cos y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \sin y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&- \cos y_7 \cos y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_7 \cos y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \cos y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \sin y_7 \cos y_5 \cos y_4 \cos x_2 \cos x_3 \sin x_4 \sin x_7 \\
&- \sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7
\end{aligned}
$$

$$+\cos y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$-\sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+\sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$-\sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\cos y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$-\cos y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$+\cos y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$+\sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+\sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+\sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$$
$$-\sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\cos y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$+\cos y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+\sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$$
$$+\sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$$
$$-\sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$
$$+\sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$$
$$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$$
$$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$-\sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$$
$$-\cos y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_2 \cos x_3 \sin x_4 \sin x_7$$
$$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$$

$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$- \cos y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$- \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_7 \sin y_5 \sin x_2 \sin y_4$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+ \sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$

$G_2[1,6] =$
$\cos y_5 \sin y_4 \cos x_2 \cos x_3 \sin x_4 \sin x_7$
$- \cos y_5 \sin y_4 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \cos y_5 \sin y_4 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$- \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$- \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$- \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$- \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$- \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$

$+ \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$- \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$- \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$- \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$- \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$+ \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$- \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$+ \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$- \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+ \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$+ \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6$
$- \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7$
$- \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6$
$- \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$+ \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$+ \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$- \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$- \sin y_5 \sin x_2 \cos y_4$

$G_2[1,7] =$
$- \cos y_6 \cos y_5 \sin x_2 \sin y_4$
$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6$
$- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7$
$- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6$
$+ \sin y_6 \cos x_2 \sin x_3 \sin x_4 \sin x_7$
$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6$
$- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6$

$$
\begin{aligned}
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_6 \sin y_5 \cos y_4 \cos x_2 \cos x_3 \sin x_4 \sin x_7 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \cos y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \sin x_6 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_2 \cos x_3 \sin x_4 \cos x_7 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \sin y_6 \cos x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \cos x_6 \\
&- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_2 \sin x_3 \sin x_4 \cos x_7 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \cos x_3 \cos x_4 \cos x_5 \sin x_6 \\
&- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \cos x_6 \\
&+ \sin y_6 \cos x_7 \cos x_2 \sin x_3 \cos x_4 \sin x_5 \sin x_6
\end{aligned}
$$

$G_2[2,1] =$

$\sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$
$+ \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$+ \sin y_6 \sin y_5 \cos y_4 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$- \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$+ \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$- \sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$- \sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$+ \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$+ \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$- \sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_6 \sin x_4 \sin x_7 \cos x_1 \cos x_3$
$- \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$- \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$
$+ \sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$

$$
\begin{aligned}
&+\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&-\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&-\sin y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&+\sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&+\sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_6 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_6 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\cos y_6 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\cos y_6 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&+\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3\\
&-\sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\cos y_6 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&-\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+\sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&-\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3
\end{aligned}
$$

$$+\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$-\sin y_6 \sin y_5 \cos y_4 \sin x_4 \sin x_7 \cos x_1 \sin x_3$$
$$-\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+\sin y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$-\sin y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$$
$$-\sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$
$$-\sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$$
$$-\sin y_6 \cos y_5 \sin x_1 \cos x_2 \sin y_4$$

$G_2[2,2] =$

$$-\sin y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\sin y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$$
$$-\sin y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+\sin y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+\sin y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$$
$$-\sin y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+\sin y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$-\sin y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+\sin y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$
$$-\sin y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\sin y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$+\sin y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+\sin y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$

$$+\cos y_7\cos y_5\sin y_4\cos y_3\cos y_2\cos y_1\sin x_4\cos x_7\sin x_1\sin x_2\sin x_3$$
$$+\cos y_7\cos y_5\sin y_4\cos y_3\cos y_2\cos y_1\sin x_7\cos x_4\cos x_5\cos x_6\cos x_1\sin x_3$$
$$-\sin y_7\sin y_6\sin y_5\cos y_4\cos y_3\cos y_2\cos y_1\sin x_7\cos x_4\cos x_5\cos x_6\sin x_1\sin x_2\cos x_3$$
$$-\sin y_7\sin y_6\sin y_5\sin y_4\cos x_7\cos x_4\sin x_5\sin x_6\cos x_1\sin x_3$$
$$+\sin y_7\cos y_6\sin y_3\sin y_2\sin y_1\sin x_7\cos x_4\sin x_5\sin x_6\cos x_1\sin x_3$$
$$+\sin y_7\cos y_6\sin y_3\sin y_2\sin y_1\sin x_4\cos x_7\sin x_1\sin x_2\cos x_3$$
$$-\sin y_7\cos y_6\sin y_3\sin y_2\sin y_1\sin x_4\cos x_7\cos x_1\sin x_3$$
$$+\sin y_7\cos y_6\sin y_3\sin y_2\cos y_1\cos x_4\cos x_5\sin x_6\sin x_1\sin x_2\sin x_3$$
$$+\sin y_7\cos y_6\sin y_3\sin y_2\cos y_1\cos x_4\sin x_5\cos x_6\sin x_1\sin x_2\cos x_3$$
$$+\sin y_7\sin y_6\sin y_5\cos y_4\sin y_3\cos y_2\sin y_1\cos x_4\cos x_5\sin x_6\cos x_1\cos x_3$$
$$-\sin y_7\sin y_6\sin y_5\sin y_4\cos x_7\cos x_4\cos x_5\cos x_6\cos x_1\cos x_3$$
$$-\sin y_7\sin y_6\sin y_5\sin y_4\sin x_4\sin x_7\cos x_1\sin x_3$$
$$-\sin y_7\cos y_6\sin y_3\cos y_2\cos y_1\sin x_7\cos x_4\cos x_5\cos x_6\sin x_1\sin x_2\cos x_3$$
$$+\sin y_7\cos y_6\sin y_3\cos y_2\cos y_1\sin x_7\cos x_4\cos x_5\cos x_6\cos x_1\sin x_3$$
$$+\sin y_7\sin y_6\sin y_5\cos y_4\cos y_3\cos y_2\sin y_1\cos x_4\cos x_5\sin x_6\sin x_1\sin x_2\cos x_3$$
$$+\sin y_7\sin y_6\sin y_5\cos y_4\sin y_3\cos y_2\sin y_1\cos x_4\cos x_5\sin x_6\sin x_1\sin x_2\sin x_3$$
$$-\sin y_7\sin y_6\sin y_5\sin y_4\cos x_7\cos x_4\cos x_5\cos x_6\sin x_1\sin x_2\sin x_3$$
$$+\sin y_7\sin y_6\sin y_5\cos y_4\cos y_3\cos y_2\cos y_1\sin x_4\cos x_7\sin x_1\sin x_2\sin x_3$$
$$-\sin y_7\cos y_6\sin y_3\sin y_2\sin y_1\sin x_7\cos x_4\sin x_5\sin x_6\sin x_1\sin x_2\cos x_3$$
$$-\sin y_7\cos y_6\sin y_3\cos y_2\sin y_1\cos x_4\sin x_5\cos x_6\cos x_1\cos x_3$$
$$-\sin y_7\cos y_6\sin y_3\cos y_2\cos y_1\sin x_7\cos x_4\sin x_5\sin x_6\sin x_1\sin x_2\sin x_3$$
$$+\sin y_7\cos y_6\sin y_3\sin y_2\sin y_1\sin x_7\cos x_4\cos x_5\cos x_6\sin x_1\sin x_2\sin x_3$$
$$-\sin y_7\cos y_6\sin y_3\cos y_2\cos y_1\sin x_7\cos x_4\sin x_5\sin x_6\cos x_1\cos x_3$$
$$+\cos y_7\cos y_5\sin y_4\sin y_3\sin y_2\sin y_1\sin x_4\cos x_7\sin x_1\sin x_2\sin x_3$$
$$-\sin y_7\cos y_6\sin y_3\cos y_2\sin y_1\cos x_4\cos x_5\sin x_6\cos x_1\sin x_3$$
$$+\sin y_7\cos y_6\cos y_3\cos y_2\sin y_1\cos x_4\sin x_5\cos x_6\cos x_1\sin x_3$$
$$-\sin y_7\cos y_6\cos y_3\cos y_2\sin y_1\cos x_4\cos x_5\sin x_6\cos x_1\cos x_3$$
$$+\sin y_7\cos y_6\sin y_3\cos y_2\cos y_1\sin x_4\cos x_7\sin x_1\sin x_2\sin x_3$$
$$-\sin y_7\cos y_6\sin y_3\cos y_2\sin y_1\cos x_4\sin x_5\cos x_6\sin x_1\sin x_2\sin x_3$$
$$+\cos y_7\cos y_5\cos y_4\cos x_7\cos x_4\cos x_5\cos x_6\cos x_1\cos x_3$$
$$-\sin y_7\sin y_6\sin y_5\cos y_4\sin y_3\sin y_2\sin y_1\sin x_7\cos x_4\sin x_5\sin x_6\sin x_1\sin x_2\sin x_3$$
$$-\sin y_7\sin y_6\sin y_5\cos y_4\cos y_3\cos y_2\sin y_1\cos x_4\cos x_5\sin x_6\cos x_1\sin x_3$$
$$+\cos y_7\cos y_5\cos y_4\cos x_7\cos x_4\cos x_5\cos x_6\sin x_1\sin x_2\sin x_3$$
$$+\sin y_7\sin y_6\sin y_5\cos y_4\sin y_3\sin y_2\sin y_1\sin x_4\cos x_7\cos x_1\cos x_3$$
$$+\sin y_7\sin y_6\sin y_5\sin y_4\sin x_4\sin x_7\sin x_1\sin x_2\cos x_3$$
$$+\sin y_7\sin y_6\sin y_5\cos y_4\sin y_3\sin y_2\cos y_1\cos x_4\cos x_5\sin x_6\cos x_1\sin x_3$$
$$+\sin y_7\cos y_6\sin y_3\cos y_2\sin y_1\cos x_4\cos x_5\sin x_6\sin x_1\sin x_2\cos x_3$$
$$-\sin y_7\sin y_6\sin y_5\cos y_4\cos y_3\cos y_2\cos y_1\sin x_7\cos x_4\sin x_5\sin x_6\cos x_1\cos x_3$$
$$+\cos y_7\cos y_5\sin y_4\cos y_3\sin y_2\sin y_1\sin x_7\cos x_4\sin x_5\sin x_6\cos x_1\sin x_3$$
$$-\cos y_7\sin y_5\cos y_4\sin y_3\cos y_2\cos y_1\cos x_4\cos x_5\sin x_6\cos x_1\cos x_3$$

$$+\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+\cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$-\cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$-\cos y_7 \sin y_5 \sin x_1 \cos x_2 \sin y_4$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+\cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$

$$+\cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$$
$$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$$
$$+\cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$$
$$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+\cos y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$

$- \cos y_7 \cos y_5 \cos y_4 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$- \cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$- \sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$- \cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$- \sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$+ \sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$- \sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$+ \cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$+ \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$- \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$
$- \sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$

$+\sin y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$+\cos y_7 \cos y_5 \cos y_4 \sin x_4 \sin x_7 \cos x_1 \sin x_3$
$-\cos y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$-\cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$-\sin y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$-\cos y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$+\sin y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$-\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+\cos y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$-\cos y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$-\sin y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$-\sin y_7 \sin y_6 \cos y_5 \sin x_1 \cos x_2 \cos y_4$
$+\sin y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$-\cos y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$-\sin y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$+\sin y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$-\sin y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$+\sin y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$-\sin y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$
$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$-\cos y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$

$G_2[2,3] =$

$-\cos y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+\cos y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$-\cos y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$-\cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$+\cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+\cos y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$

$- \sin y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$+ \cos y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$+ \cos y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$+ \cos y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$- \cos y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \cos y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$- \cos y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$+ \cos y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$- \cos y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$

$$
\begin{aligned}
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3\\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3\\
&+ \sin y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&- \sin y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&+ \sin y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&+ \sin y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+ \sin y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+ \sin y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&+ \sin y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&- \sin y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+ \cos y_7 \cos y_6 \cos y_5 \sin x_1 \cos x_2 \cos y_4\\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&+ \cos y_7 \cos y_6 \sin y_5 \sin y_4 \sin x_4 \sin x_7 \cos x_1 \sin x_3
\end{aligned}
$$

$$
\begin{aligned}
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \sin y_4 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \sin y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \sin y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \sin y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \sin y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \sin y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \sin y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&- \cos y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \sin y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \sin y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \sin y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \cos y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3
\end{aligned}
$$

$$
\begin{aligned}
&- \sin y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \sin y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \sin y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&+ \sin y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3 \\
&+ \cos y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&- \sin y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \sin y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \sin y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \sin y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&+ \sin y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3
\end{aligned}
$$

$-\cos y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$

$G_2[2,4] =$

$$
\begin{aligned}
&-\cos y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3\\
&+\sin y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&+\cos y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&-\cos y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&-\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&-\cos y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\cos y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3\\
&+\cos y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_7 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_7 \cos y_6 \cos y_5 \sin x_1 \cos x_2 \cos y_4\\
&+\cos y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\cos y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&+\cos y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\sin y_7 \cos y_6 \sin y_5 \sin y_4 \sin x_4 \sin x_7 \cos x_1 \sin x_3\\
&+\sin y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\sin y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\sin y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&-\sin y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\sin y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_7 \sin y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\sin y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \sin y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3
\end{aligned}
$$

$$
\begin{aligned}
&+ \sin y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \sin y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \sin y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \sin y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \sin y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \sin y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \sin y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \sin y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3 \\
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \sin y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3
\end{aligned}
$$

$$- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$- \sin y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+ \sin y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$$
$$- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+ \cos y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+ \cos y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$$
$$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \cos y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$- \sin y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$- \sin y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+ \sin y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+ \sin y_7 \sin y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$- \sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$$
$$- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$
$$- \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+ \sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$

$$
\begin{aligned}
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \sin y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\cos y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&-\sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \cos y_6 \sin y_5 \sin y_4 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_7 \cos y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&-\sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3\\
&-\sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&-\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&-\cos y_7 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\cos y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&+\sin y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_7 \cos y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\cos y_7 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&+\sin y_7 \sin y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\cos y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \cos y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&-\cos y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\cos y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3
\end{aligned}
$$

$- \cos y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$- \sin y_7 \sin y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$
$+ \cos y_7 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \cos y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$

$G_2[2,5] =$

$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$- \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$- \sin y_7 \cos y_5 \cos y_4 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$+ \sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$
$- \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$

$$
\begin{aligned}
&-\sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&+\cos y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&+\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&+\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&-\cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&+\cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&+\cos y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\cos y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&-\cos y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&+\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3
\end{aligned}
$$

$-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$
$+\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$-\cos y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$-\cos y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$-\cos y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$-\cos y_7 \sin y_6 \sin y_5 \sin y_4 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \cos x_3$
$+\sin y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$+\sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$-\sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$-\sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$-\sin y_7 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3$
$+\sin y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$-\sin y_7 \cos y_5 \cos y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$-\cos y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$-\cos y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$
$+\cos y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$
$-\cos y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$+\sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$-\sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$+\sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$-\sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$+\sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+\sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$
$-\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$+\cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$-\sin y_7 \sin y_5 \sin x_1 \cos x_2 \sin y_4$
$-\cos y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+\cos y_7 \cos y_6 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$-\cos y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$+\cos y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$+\cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$-\cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$+\cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$

$$
\begin{aligned}
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \cos y_6 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3 \\
&+ \cos y_7 \cos y_6 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin x_1 \cos x_2 \cos y_4 \\
&+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3 \\
&- \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&- \cos y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_7 \cos y_6 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3
\end{aligned}
$$

$$
\begin{aligned}
&- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&- \cos y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&+ \cos y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&- \cos y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&+ \cos y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&- \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&- \sin y_7 \cos y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&- \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+ \sin y_7 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+ \cos y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&- \cos y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \cos y_6 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \cos y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+ \cos y_7 \cos y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&+ \cos y_7 \cos y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&+ \cos y_7 \cos y_6 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3
\end{aligned}
$$

$- \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$
$+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \cos y_7 \sin y_6 \cos y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_6 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \sin y_7 \cos y_5 \cos y_4 \sin x_4 \sin x_7 \cos x_1 \sin x_3$
$+ \cos y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$+ \cos y_7 \sin y_6 \sin y_5 \sin y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$- \cos y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \sin y_6 \sin y_5 \sin y_4 \sin x_4 \sin x_7 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$
$- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \cos y_7 \sin y_6 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$
$- \sin y_7 \sin y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$
$+ \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \sin y_7 \cos y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$
$+ \cos y_7 \cos y_6 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$

$G_2[2,6] =$

$- \cos y_5 \sin y_4 \sin x_4 \sin x_7 \cos x_1 \sin x_3$
$- \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$
$- \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$
$+ \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$
$- \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$

$$
\begin{aligned}
&-\cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\cos y_5 \sin y_4 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \cos x_3\\
&+\sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&-\cos y_5 \sin y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\cos y_5 \sin y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&+\sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&+\sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+\cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3\\
&-\cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3\\
&+\cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&-\cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+\cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3\\
&+\cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&+\cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3\\
&+\cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3\\
&+\cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3\\
&-\cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3\\
&-\sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3\\
&+\sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3\\
&-\sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3\\
&+\cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3\\
&+\sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3\\
&+\cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3\\
&-\sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3
\end{aligned}
$$

$$
\begin{aligned}
&- \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&+ \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_5 \sin x_1 \cos x_2 \cos y_4 \\
&- \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3 \\
&+ \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_5 \sin y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \cos y_5 \sin y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3
\end{aligned}
$$

$G_2[2,7] =$

$$
\begin{aligned}
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&- \sin y_6 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \sin y_6 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&+ \cos y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \sin y_6 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_6 \sin y_5 \cos y_4 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_6 \sin y_5 \cos y_4 \sin x_4 \sin x_7 \cos x_1 \sin x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_6 \cos y_5 \sin x_1 \cos x_2 \sin y_4 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \sin y_6 \sin x_4 \sin x_7 \cos x_1 \cos x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3
\end{aligned}
$$

$$+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$- \cos y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+ \cos y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \cos y_6 \sin y_5 \cos y_4 \cos x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+ \sin y_6 \cos x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \sin y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$+ \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \cos x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$$
$$- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \cos x_1 \sin x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3$$
$$- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \sin x_3$$
$$- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \cos x_3$$
$$- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \cos x_1 \sin x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \sin x_3$$
$$+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \cos x_3$$
$$- \cos y_6 \sin y_5 \sin y_4 \sin y_3 \cos y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3$$

$$
\begin{aligned}
&+ \sin y_6 \sin x_4 \sin x_7 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \cos x_1 \cos x_3 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \cos x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \sin y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \cos x_3 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \sin x_1 \sin x_2 \sin x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \cos x_1 \sin x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_4 \cos x_7 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_6 \sin y_5 \sin y_4 \sin y_3 \sin y_2 \cos y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3 \\
&- \cos y_6 \cos y_5 \cos y_4 \cos y_3 \cos y_2 \sin y_1 \sin x_7 \cos x_4 \cos x_5 \cos x_6 \sin x_1 \sin x_2 \cos x_3 \\
&+ \cos y_6 \cos y_5 \cos y_4 \cos y_3 \sin y_2 \cos y_1 \sin x_7 \cos x_4 \sin x_5 \sin x_6 \cos x_1 \sin x_3 \\
&- \cos y_6 \sin y_5 \sin y_4 \cos y_3 \cos y_2 \sin y_1 \cos x_4 \sin x_5 \cos x_6 \sin x_1 \sin x_2 \sin x_3
\end{aligned}
$$

Bibliography

[1] Bobby S. Acharya, Xenia De La Ossa, and Sergei Gukov, *G-flux, supersymmetry and* Spin(7) *manifolds*, J. High Energy Phys. (2002), no. 9, No. 047, 13. MR 2004b:81136

[2] J. F. Adams, *Lectures on exceptional Lie groups*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996, With a foreword by J. Peter May, Edited by Zafer Mahmud and Mamoru Mimura. MR MR1428422 (98b:22001)

[3] John C. Baez, *The octonions*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 2, 145–205 (electronic). MR 2003f:17003

[4] Robert L. Bryant, *Metrics with holonomy G_2 or* Spin(7), Workshop Bonn 1984 (Bonn, 1984), Lecture Notes in Math., vol. 1111, Springer, Berlin, 1985, pp. 269–277. MR 87a:53082

[5] ______, *Metrics with exceptional holonomy*, Ann. of Math. (2) **126** (1987), no. 3, 525–576. MR 89b:53084

[6] ______, *Some remarks on g_2-structures*, 2003.

[7] Robert L. Bryant and Simon M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), no. 3, 829–850. MR 90i:53055

[8] Simon Chiossi and Simon Salamon, *The intrinsic torsion of* SU(3) *and G_2 structures*, Differential geometry, Valencia, 2001, World Sci. Publishing, River Edge, NJ, 2002, pp. 115–133. MR 2003g:53030

[9] Morton L. Curtis, *Matrix groups*, second ed., Universitext, Springer-Verlag, New York, 1984. MR MR766229 (85k:22001)

[10] M. Fernández and A. Gray, *Riemannian manifolds with structure group* G_2, Ann. Mat. Pura Appl. (4) **132** (1982), 19–45 (1983). MR 84e:53056

[11] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR MR1153249 (93a:20069)

[12] V. V. Gorbatsevich, A. L. Onishchik, and E. B. Vinberg, *Foundations of Lie theory and Lie transformation groups*, Springer-Verlag, Berlin, 1997. MR MR1631937 (99c:22009)

[13] Brian C. Hall, *Lie groups, Lie algebras, and representations*, Graduate Texts in Mathematics, vol. 222, Springer-Verlag, New York, 2003, An elementary introduction. MR MR1997306 (2004i:22001)

[14] F. Reese Harvey, *Spinors and calibrations*, Perspectives in Mathematics, vol. 9, Academic Press Inc., Boston, MA, 1990. MR MR1045637 (91e:53056)

[15] Reese Harvey and H. Blaine Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157. MR 85i:53058

[16] Bo Ju Jiang and Chia-Kuei Peng (eds.), *Differential geometry and topology*, Lecture Notes in Mathematics, vol. 1369, Berlin, Springer-Verlag, 1989. MR MR1001177 (90a:53002)

[17] Dominic D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. MR MR1787733 (2001k:53093)

[18] Naichung Conan Leung, *Riemannian geometry over different normed division algebras*, J. Differential Geom. **61** (2002), no. 2, 289–333. MR 2004e:53074

[19] Mikio Nakahara, *Geometry, topology and physics*, second ed., Graduate Student Series in Physics, Institute of Physics, Bristol, 2003. MR 2004e:58001

[20] N. J. Wildberger, *A combinatorial construction of* G_2, J. Lie Theory **13** (2003), no. 1, 155–165. MR MR1958579 (2004d:17013)