Blurring Invariants

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April 30, 2006

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Senior Thesis
How can we tell which picture is the blurred version of Professor Gu?
One way is to find a function $h$ that is \textit{invariant} under blurring.
Infinite, One-dimensional, Continuous, Black and White Images

The images we will be considering are continuous images. In particular, we will consider functions $f_0 : \mathbb{R} \rightarrow [0, 1]$.

- Functions like this can be seen as coloring the real line.
- $f_0(x) = 1$ means that $x$ is colored white, while $f_0(x) = 0$ means $x$ is colored black. If $f_0(x)$ is something in between, then it will be colored the corresponding shade of grey.
So things will always converge, we insist $f_0(x)$ is small at $\infty$ and $-\infty$. Specifically,

$$\int_{-\infty}^{\infty} e^{sx} f_0(x) dx$$

converges for some $s > 0$. 
The Gaussian function (the bell shaped curve) with parameter $t$ is given by

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$$
Let $f_t$ denote the image that is the Gaussian blur of image $f_0$ with parameter $t$, where

$$f_t(x) = \int_{-\infty}^{\infty} g_t(s-x)f_0(s)ds$$

Notice $f_t(x)$ just the convolution of $f_0$ and $g_t$, usually denoted $f_0 * g_t$.

What if we apply a Gaussian blur to a Gaussian blur? Amazingly, we get another Gaussian blur. In fact

$$(f_0 * g_{t_0}) * g_{t_1} = f_0 * g_{t_0+t_1}$$

This means that the operation of Gaussian blurring is a semigroup.
Let $h$ be a function from infinite, one-dimensional, continuous, black and white images to the reals. We say that $h$ is an invariant under Gaussian blurring if

$$h(f_t) = \text{a constant for all } t$$

Which is equivalent to saying

$$\frac{d}{dt} h(f_t) = 0.$$ 

We say that $h$ is functionally independent from $h'$ if there is no function $u : \mathbb{R} \to \mathbb{R}$ such that

$$u(h(f)) = h'(f)$$

for all $f$. 


Using Intuition

Blurring does not lighten or darken an image. Measuring the total lightness or darkness is accomplished by

\[ h_0(f) = \int_{-\infty}^{\infty} f(x) \, dx. \]

Sure enough, \( h_0 \) is an invariant!

\[
\begin{align*}
  h_0 \left( \int_{-\infty}^{\infty} g_t(s-x)f_0(s) \, ds \right) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_t(s-x)f_0(s) \, ds \, dx \\
  &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_t(s-x)f_0(s) \, dx \, ds \\
  &= \int_{-\infty}^{\infty} f_0(s) \int_{-\infty}^{\infty} g_t(s-x) \, dx \, ds \\
  &= \int_{-\infty}^{\infty} f_0(s)(1) \, ds \\
  &= \text{a constant.}
\end{align*}
\]
Blurring does not seem to move the “center of mass” of the image. Measuring this is accomplished by

$$ h_1(f) = \int_{-\infty}^{\infty} xf(x) \, dx. $$

Sure enough, $h_1$ is an invariant too!
Define

\[ h_i(f) = \int_{-\infty}^{\infty} x^i f(x) \, dx. \]

Maybe \( h_i(f) \) is invariant for all \( i \)?
Define

\[ h_i(f) = \int_{-\infty}^{\infty} x^i f(x) dx. \]

Maybe \( h_i(f) \) is invariant for all \( i \)? **Nope ...**
Generalizing

It fails for the case $h_2(f)$. After some computation, we get

\[
h_2(f_t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 g_t(s - x) f_0(s) ds dx
\]

\[
= \int_{-\infty}^{\infty} s^2 f_0(s) ds + t \int_{-\infty}^{\infty} f_0(s) ds
\]

\[
= h_2(f_0) + th_0(f_0)
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It depends on $t$...
Table of values for $h_i(f_t)$:

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Perhaps we can take linear combinations of these to come up with more invariants?
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**Yep** — look at $h_0(f)h_3(f) - 3h_1(f)h_2(f)$.

The $t$ terms cancel, and we’re left with constant terms.
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Some more invariants:

\[ h_0 h_4 - 3[h_2]^2 \]

\[ 3h_1 h_5 - 5[h_3]^2 \]

\[ [h_0]^2 h_6 - 15h_0 h_2 h_4 + 30[h_2]^3 \]
**Conjecture** There are an infinite number of functionally independent invariants under Gaussian blurring of an infinite continuous one-dimensional image of the form

\[ \alpha(h_0, h_1, \ldots, h_k) \]

for some \( k \) and some polynomial \( \alpha : \mathbb{R}^k \to \mathbb{R} \).
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- Prove that there exist invariants of the form $\alpha(h_0, \ldots, h_k)$ for infinitely many $k$. Proving this probably will involve induction, using invariants for lower values of $k$ to find invariants for higher values of $k$. 
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A proof might proceed in these two parts:

- Prove that there exist invariants of the form \( \alpha(h_0, \ldots, h_k) \) for infinitely many \( k \). Proving this probably will involve induction, using invariants for lower values of \( k \) to find invariants for higher values of \( k \).
- Prove that \( \alpha(h_0, \ldots, h_{k_1}) \) is functionally independent from \( \alpha(h_0, \ldots, h_{k_2}) \) for \( k_2 > k_1 \). This involves finding pairs of images \( f \) and \( f' \) such that the first invariant cannot distinguish between them, but the second can.
Future Work

• Prove the conjecture!

• Explore invariants not in the form $\alpha(h_0, \ldots, h_k)$. Are there any? Can they be completely characterized?

• Extend ideas to other group-like transformations on continuous images
I’d like to thank Professor Gu for all her wonderful ideas and contributions to my thesis.

Bibliography