Research Proposal:  
Matrix Perturbation Theory  

Tyler Seacrest  

Faculty Advisor: Professor Weiqing Gu  

1 Introduction  

I plan on studying how small perturbations in a matrix can affect properties of the matrix. This has many applications, like in physics. Even modern physics such as M-theory uses matrices and perturbation techniques [3]. Many other physical systems many physical problems can be modeled with systems of differential equations. Many of these systems are linear systems, and if not, can be approximated by linear systems. At this point, the model is reduced down to something like

\[ x' = Ax \]

for some possibly time-dependent matrix \( A \) and time-dependent vector \( x \). The behavior of a model of this form depends heavily on the eigenvalues and eigenvectors of \( A \). One question that often arises is how small perturbations in the model affect the behavior of the model. This translates directly into how small perturbations in matrix \( A \) affect its eigenvalues and eigenvectors.

I propose finding the relationship between perturbations and the changes in eigenvalues and eigenvectors for specific classes of matrices.

2 Proposed Research  

There are many types of matrices that deserve study. One example that I would definitely like to examine are symmetric matrices. They necessarily have a full linearly independent set of eigenvectors, and are thus diagonalizable. In the \( 3 \times 3 \) case, symmetric matrices have an interesting geometric interpretation in quadratic forms, graphing the equation \( X^TAX = 1 \). The eigenvalues of the matrix determine the shape of the three dimensional surface (see figure 1).

Since the eigenvalues determine the surface, the way perturbations in the matrix affect the eigenvalues will give us insight into the shape of the surface. It will be especially interesting to study matrices where small perturbations change the surface from one topological type to another.
Figure 1: This shows three different behaviors of quadratic forms. The first corresponds to three positive eigenvalues, the second with one negative and two positive eigenvalues, and the third with one positive and two negative eigenvalues.

Another type of interesting matrix is the set of stochastic matrices, where each entry is taken from the closed interval $[0, 1]$ and the sum of each column is one. These are used in Markov chains, and the steady state vectors in a Markov chain are the eigenvectors associated with $\lambda = 1$. Therefore, the manner in which small perturbations in the matrix affect the eigenvectors is useful for determining the behavior of the Markov chain.

A perturbation can be a “small” matrix $P$ such that $A + P$ is not far from matrix $A$ in some sense. To make $P$ small, it requires some sort of norm put on $P$ such that the norm of $P$ is less than some $\epsilon$. A first step in research would be finding an upper bound on the amount the eigenvalues and eigenvectors change in terms of $\epsilon$. Many different norms may be used, including the $L^2$ norm, infinite norm, Frobenius Norm, Hilbert-Schmidt Norm, and others [1].

3 Prior Research

The research will be based off of work done in Professor Gu’s Advanced Linear Algebra class.

References
