A Tiling Interpretation of q-Binomial Coefficients

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Abstract

I have developed a tiling interpretation of the q-binomial coefficients. The aim of this thesis is to apply this combinatorial interpretation to a variety of q-identities to provide straightforward combinatorial proofs. The range of identities I present include q-multinomial identities, alternating sum identities and congruences.

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Chapter 1

Introduction to q-Binomial Coefficients

1.1 Research goals

The binomial coefficients are one of the essential building blocks of enumerative combinatorics. A great deal of research has gone into understanding them both algebraically and combinatorially, and there are a wealth of binomial identities with both algebraic and combinatorial proofs. My work this semester has focused on a generalized version of the binomial coefficients: the q-binomial coefficients. These generalized binomial coefficients appear naturally in the studies of integer partitions and hypergeometric series, and their properties have been examined primarily in these contexts. However, previous study of q-binomial coefficients has tended towards an algebraic viewpoint. My goal for this project has been to provide combinatorial interpretations for q-binomial identities. This includes both giving combinatorial proofs for known q-identities and using a combinatorial understanding of standard binomial identities to find and prove q-analogues.

1.2 Notation and Basic Theory

There are several equivalent algebraic definitions for the q-binomial coefficients. From a combinatorics perspective, it makes sense to start by generalizing the natural numbers and work our way up to binomial coefficients. For a natural number k, we define its q-version $[k]_q$ as a polynomial in q by:

$$[k]_q \equiv 1 + q + q^2 + \cdots + q^{k-1}$$

Then we define the q-factorial in the natural way:

$$[k]_q! \equiv [k]_q[k-1]_q[k-2]_q \cdots [1]_q$$

and we define the q-binomial coefficient in the natural way as well:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

Note that if we let $q = 1$, we get

$$[k]_1 = 1 + 1 + 1 + \ldots + 1 = k$$

and it follows that

$$[k]_1! = k!$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

A standard combinatorial interpretation for $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is that it counts the number of partitions that will fit into a box of size $k \times (n-k)$, weighted by the size of the partition. For example, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$ evaluates to $q^4 + q^3 + 2q^2 + q + 1$. This tells us that there are a total of six distinct integer partitions which fit into a 2×2 box and that these include one partition of the number 4, one of the number 3, two of the number 2, one of 1, and one of 0. Indeed, if we look for all the partitions that fit into a 2×2 box, we find $(2,2), (2,1), (2), (1,1), (1)$ and $()$. These are precisely the number and types of partitions predicted by $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q$. (See Figure 1.1).

In practice, combinatorial proofs using this partition-in-a-box interpretation often make use of an associated lattice path interpretation. Each partition that fits into a box of size $k \times (n-k)$ is uniquely determined by a lattice path from $(0,0)$ to $(k, n-k)$ and we simply take the weight of the lattice path to be the weight of its associated partition. Figure 1.1 shows the partition/lattice path interpretation for $n = 4, k = 2$.

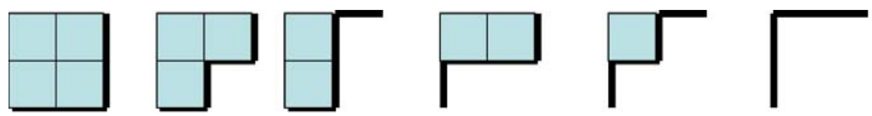


Figure 1.1: The six lattice paths from (0,0) to (2,2). The exponent on q in the weight of each path is given by counting the number of boxes which fit above and to the left of the lattice path.

1.3 q-Binomial coefficients in linear algebra

The partition-in-a-box interpretation gives the q-binomial coefficients a natural application to the theory of integer partitions. However, they show up in other fields as well. For example, let V be the n-dimensional vector space over the finite field F_q. (That is, V is the set of all vectors of length n whose elements are in F_q.) We might naturally ask the question, "How many k-dimensional subspaces does V have?"

To answer this question, note that the number of k-dimensional subspaces of V multiplied by the number of *ordered* bases for each subspace gives us the number of *ordered* bases for all k-dimensional subspaces of V.

Now, once we have picked a k-dimensional subspace, how many ways can we choose an ordered basis? For the first basis vector, we may choose any element of the subspace except $\mathbf{0}$. Thus, we have $(q^k - 1)$ choices. For the second basis vector, we may choose any element of the subspace except q elements spanned by our first basis vector for a total of $(q^k - q)$ choices. We proceed to pick all k of our vectors in this fashion. This gives us the number of ordered bases for each k-dimensional subspace as:

$$(q^k - 1)(q^k - q)(q^k - q^2)\dots(q^k - q^{k-1})$$

On the other hand, we also need to count the number of ordered bases for all k-dimensional subspaces. We do this in a similar fashion. For our first basis vector, we may choose anything in the vector space except $\mathbf{0}$, for a total of $q^n - 1$ choices. For our second basis vector we may choose anything except the q elements spanned by our first basis vector, for a total of $q^n - q$ choices. Proceeding in this fashion, we find that the total number of ordered bases for all k-dimensional subspaces is

$$(q^n - 1)(q^n - q)(q^n - q^2)\dots(q^n - q^{k-1})$$

Thus, the number of k-dimensional subspaces of V is the quotient of these expressions:

$$\frac{(q^n-1)(q^n-q)(q^n-q^2)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)\dots(q^k-q^{k-1})}$$

Factoring out all the factors of q from the top and bottom of this expression gives

$$\frac{q^{\binom{k}{2}}(q^n-1)(q^{n-1}-1)(q^{n-2}-1)\dots(q^{n-k}-1)}{q^{\binom{k}{2}}(q^k-1)(q^{k-1}-1)(q^{k-2}-1)\dots(q-1)}$$

and further factorization by pulling out a $(q-1)$ from each term on the top and bottom gives

$$\frac{(q-1)^k[n]_q[n-1]_q[n-2]_q\dots[n-k+1]_q}{(q-1)^k[k]_q[k-1]_q[k-2]_q\dots[1]_q}$$

That is, the number of k-dimensional subspaces of V is

$$\frac{[n]_q!}{[k]_q![n-k]_q!} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Thus, the q-binomial coefficients can arise naturally in questions of enumeration in linear algebra.

Chapter 2

A Tiling Interpretation of the q-Binomial Coefficients

2.1 The Tiling Interpretation

The standard binomial coefficient $\binom{n}{k}$ counts, among other things, the number of ways to tile a board of length n using k green squares and $n-k$ red squares. This interpretation can be extended to the q-binomial coefficients by assigning each tiling a weight of the form q^w. Let $\mathcal{T}_{n,k}$ be the set of all tilings of an n-board using exactly k green squares and $n-k$ red squares. Also let q^{w_T} be the weight of tiling T. For each $T \in \mathcal{T}_{n,k}$, we calculate w_T as follows:

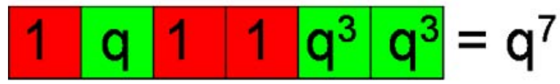
- Assign a weight to each individual square in the tiling. A red square always receives a weight of 1. A green square has weight q^s where s is equal to the number of red squares to the left of that green square in the tiling.

- Calculate w_T by multiplying the weight q^s of all the green squares (or equivalently, the weight of *all* of the squares.)

For example, the weight of the tiling *rgrrgg* is $q^{1+3+3} = q^7$, as demonstrated in Figure 2.1.

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is created by summing the weights of all these tilings. That is,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{T \in \mathcal{T}_{n,k}} q^{w_T}$$



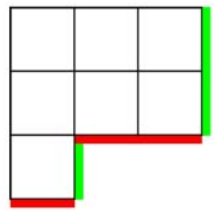


Figure 2.1: Here we see a sample tiling and its associated lattice path. Note that the tiling receives weight q^7 and the lattice path corresponds to a partition of the number 7.

Since $\lim_{q\to 1} q^w = 1$ for any $w \in \mathbb{N}$, we see that $\lim_{q\to 1} \sum_{T\in\mathcal{T}_{n,k}} q^{w_T} = \sum_{T\in\mathcal{T}_{n,k}} \lim_{q\to 1} q^{w_T} = \sum_{T\in\mathcal{T}_{n,k}} 1$, which is just the standard binomial coefficient $\binom{n}{k}$.

Note that there is an obvious bijection between this tiling interpretation and the boxed partition interpretation of the q-binomial coefficients. To each n-tiling with k green squares, create an associated lattice path from $(0,0)$ to $(n-k,k)$ by letting each green tile represent a move one unit up and each red square represent a move one unit right. (See Figure 2.1)

This bijection clearly gives the same number of tilings and boxed partitions. It just remains to show that the weight of the tiling and its associated lattice path are the same. To see this, note that we can calculate the weight of the lattice path by summing one row at a time. That is, since each row corresponds to an up move, for each up move, the weight of that row is given by the number of preceding right moves in the path. This is precisely how we calculate the weight of our tilings, since the weight of each green tile is determined by the number of red tiles before it. Therefore, the bijection between partitions/lattice paths and tilings is weight-preserving.

Hence, since $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts the number of partitions which will fit into a box of size $k \times (n-k)$ weighted by the size of the partition, it also counts the number of n-tilings with k green squares and $n-k$ red squares weighted as described above.

Since the subscript on the q-binomial coefficient will almost always be q, from this point forward we will simply write $\begin{bmatrix} n \\ k \end{bmatrix}$ for $\begin{bmatrix} n \\ k \end{bmatrix}_q$. The missing subscript should be taken to be q unless otherwise specified.

2.2 Some q-Binomial identities with simple proofs under this interpretation

We will now present a selection of q-identities whose proofs are straightforward using the tiling interpretation for the q-binomial coefficients. The identities in this section are taken from George Andrews and Kimmo Eriksson's book *Integer Partitions* [2], where they are presented in slightly different form. Their proofs are given combinatorially via the partition-in-a-box interpretation of the q-binomial coefficients. I provide them here both for reference and to demonstrate their proofs using the tiling interpretation. All of these identities have well-known binomial coefficient analogs when $q = 1$.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} \tag{2.1}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q^{n-k} \tag{2.2}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \tag{2.3}$$

$$\begin{bmatrix} 2n \\ n \end{bmatrix} = \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}^2 \tag{2.4}$$

Proof of (2.1):

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$$

We not only need a bijection between the tilings on the left and right side of the equation, but a *weight-preserving* bijection. That is, we must find a way to map each tiling of length n with k green squares to a tiling of length n with $n-k$ green squares that has the same weight. A first guess would be to simply toggle each square between red and green. This gives a bijection, but not necessarily a weight-preserving one. (For example, the tiling

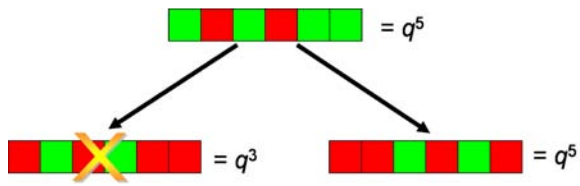


Figure 2.2: If we take the top tiling and reverse it, we obtain a tiling of different weight. However, if we instead toggle the color of each tile and reverse, we obtain a new tiling of the same weight.

grgrgg which has weight q^5 would get mapped to *rgrgrr* which has weight q^3, as shown in Figure 2.2.) Instead, we choose the bijection given by toggling each tile between red and green *and* reversing the order of the tiling. This works because counting the number of red tiles before each green is equivalent to counting the number of green tiles after each red.

Proof of (2.2):

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q^{n-k}$$

Question: What is the total weight of the tilings of length n using k green squares and $n-k$ red squares?
Answer 1: By definition,

$$\begin{bmatrix} n \\ k \end{bmatrix}.$$

Answer 2: Consider the color of the last tile.

The total weight of the tilings whose last tile is *red* is simply the weight of all tilings of length $n-1$ with k green squares multiplied by the weight of the final red tile. Since the weight of the final red tile must be 1, we have a total weight of $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ from tilings of this form.

On the other hand, if the last tile is *green*, then we again have the sum of the weights of the tilings of the first $n-1$ positions with $k-1$ green squares multiplied by the weight of the final green tile. The final green tile always has weight q^{n-k} because it has $(n-k)$ red squares before it.

Thus, the total weight of all tilings of length n with k green squares is

$$\begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

as desired.

Proof of (2.3):

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

Question: What is the total weight of the tilings of length n using k green squares and $n-k$ red squares?
Answer 1: By definition,

$$\begin{bmatrix} n \\ k \end{bmatrix}.$$

Answer 2: Consider the color of the first tile.

If it is green, then it contributes a weight of 1 to each tiling. Thus, our answer is just the sum of the weights of the tilings of the last $n-1$ positions with $k-1$ green squares. That is, $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.

On the other hand, if the first tile is red, then we know it has weight 1 itself, but it also contributes a weight of q to each of the following k green squares. Thus, the total weight contribution of the initial red is q^k. Since we also have $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ as the weight of the tilings of the last $n-1$ positions with k green squares, the total weight of these tilings is $q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$.

Therefore, the total number of weighted tilings is

$$\begin{bmatrix} n-1 \\ k \end{bmatrix} q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

as desired.

Proof of (2.4):

$$\begin{bmatrix} 2n \\ n \end{bmatrix} = \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}^2$$

Question: What is the sum of the weights of the tilings of length $2n$ using n green squares and n red squares?
Answer 1: By definition,

$$\begin{bmatrix} 2n \\ n \end{bmatrix}.$$

Answer 2: Consider the number of red squares out of the first n. Say j of the first n squares are red. Then the sum of the weights of the tilings of the first n positions is $\begin{bmatrix} n \\ n-j \end{bmatrix}$. If we temporarily ignore the first n positions of the tiling, the sum of the weights of the tilings of the last n positions is $\begin{bmatrix} n \\ j \end{bmatrix}$. However, the weight of the tiling covering the last n positions is adjusted from $\begin{bmatrix} n \\ j \end{bmatrix}$ because it is preceded by some number of red squares. In particular, since the first n tiles contained j red squares, each green square in the second half gets an additional weight of q^j. Thus, since there are j green squares among the last n positions, we have undercounted the weight of the tiling as a whole by a factor of $(q^j)^j = q^{j^2}$. That is, for a fixed j, the total weight of the tilings is

$$q^{j^2} \begin{bmatrix} n \\ n-j \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix}.$$

Finally, the weight of all desired tilings is the sum over j. That is, our answer is

$$\sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ n-j \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix}$$

Hence, by applying Equation (2.1), we get the desired identity:

$$\begin{bmatrix} 2n \\ n \end{bmatrix} = \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}^2$$

2.3 The q-Binomial Theorem

The standard binomial theorem states

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

We can prove the binomial theorem by choosing either 1 or x from each of the $(1+x)$ terms on the left. The number of ways to choose exactly k of the xs and therefore obtain an x^k coefficient is $\binom{n}{k}$.

We will now prove the q-binomial theorem in a similar way.
Theorem:

$$\prod_{j=0}^{n-1} (1+xq^j) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

Proof:
Question: What is the sum of the weights of the tilings of an n-board with $n-k$ red and k green squares?

Answer 1: By definition, $\begin{bmatrix} n \\ k \end{bmatrix}$. That is, the x^k coefficient of $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$.

Answer 2: From the product $\prod_{j=0}^{n} (1+xq^j)$, construct a tiling by choosing precisely $(n-k)$ ones and k non-one terms to obtain an x^k coefficient. Each 1 term chosen represents a red square whereas each xq^j term represents a green square. However, choosing terms in this manner creates a larger power of q than the one which is given by the corresponding tiling, so we now need to figure out how much this technique overestimates the exponent on q.

Recall that for each green tile selected, the exponent on q should be the number of red tiles preceding it. Thus, for the first xq^j term chosen, the previous j terms chosen must have been 1s, so q^j is the correct weight of that green square. However, for the second xq^j term selected, $j-1$ of the previous j selections were red squares and one was green, so the correct weight of this square is only q^{j-1}, not the q^j we actually multiplied in. Similarly, the third xq^j selected represents a green square with weight q^{j-2} and we have overcounted by q^2. All told, we have overcounted by a factor of $q^{0+1+2+\ldots+(k-1)} = q^{\binom{k}{2}}$.

That is, the answer to our question is the x^k coefficient of $\prod_{j=0}^{n} (1+xq^j)/q^{\binom{k}{2}}$

Therefore, since they have the same coefficient on x^k for all k,

$$\prod_{j=0}^{n-1} (1 + xq^j) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

as desired.

Chapter 3

q-Identities Requiring More Advanced Techniques

3.1 The Vandermonde Convolution

The Vandermonde convolution is a commonly used combinatorial identity. The identity is often written as

$$\binom{n}{m} = \sum_k \binom{p}{k}\binom{n-p}{m-k}. \tag{3.1}$$

We will present a proof sketch for this identity and then generalize to a q-identity.

Proof sketch:

Question: How many ways are there to tile an n-board with m green tiles and $n-m$ red tiles?

Answer 1: By definition, $\binom{n}{m}$.

Answer 2: Consider the number of green tiles in the first p positions. (Call this number k.) Our answer is $\binom{p}{k}\binom{n-p}{m-k}$ summed over all values of k.

The proof of the following q-identity proceeds along the same lines. **Claim:**

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} p \\ k \end{bmatrix} \begin{bmatrix} n-p \\ m-k \end{bmatrix} q^{(p-k)(m-k)} \tag{3.2}$$

Proof:

Question: What is the sum of the weights of the tilings of an n-board with m green tiles and $n-m$ red tiles?

Answer 1: By definition, $\begin{bmatrix} n \\ m \end{bmatrix}$.

Answer 2: Consider the number of green tiles in the first p positions. (Call this number k.) For a fixed k, the sum of the weights of the tilings of the first p positions is $\begin{bmatrix} p \\ k \end{bmatrix}$. If we ignore the first p positions, the sum of the weights of the tilings on the last $n-p$ positions is $\begin{bmatrix} n-p \\ m-k \end{bmatrix}$. However, we must additionally adjust this weight to account for the fact that this section of the tiling is preceded by a tiling containing $p-k$ red tiles. That is, each of the $m-k$ green tiles in the second section gets extra weight $p-k$ due to the red tiles in the first section, so we must adjust our weight up by $q^{(p-k)(m-k)}$. Therefore, the total weight of the tilings we're looking for is

$$\sum_k \begin{bmatrix} p \\ k \end{bmatrix} \begin{bmatrix} n-p \\ m-k \end{bmatrix} q^{(p-k)(m-k)}$$

as desired.

Note: if we instead consider the number of green tiles in the *last* p positions, we get the similar identity:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} p \\ k \end{bmatrix} \begin{bmatrix} n-p \\ m-k \end{bmatrix} q^{(n-p-m+k)k} \tag{3.3}$$

3.2 The Gaussian Formula

This is another formula taken from Andrews and Eriksson's *Integer Partitions* [2]. In their book, it is proved by substituting in several recurrence relations. We will present instead a combinatorial proof using tilings.

Theorem:

$$\sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = \begin{cases} 0 & \text{if n is odd} \\ (1-q)(1-q^3)(1-q^5)\dots(1-q^{n-1}) & \text{if n is even} \end{cases} \tag{3.4}$$

Proof: For odd n, the proof is simple. For each tiling T, create the tiling T' by reversing the order of T and toggling the color of each tile. This procedure preserves the weight of T, but changes the parity of the number of green tiles. (Note that if n were even, toggling the color of each tile would preserve the parity of the number of green squares, rather than changing it.) Furthermore, if we perform this procedure on T', we will get T back. Thus, we have a sign-reversing involution, so the positive and negative terms of our sum cancel out and $\sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = 0$

The case where n is even is more complicated. If we can show combinatorially that

$$\sum_{j=0}^{n+2} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = (1 - q^{n+1}) \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}$$

then we may proceed by induction. To show this, split the tilings of an $(n+2)$-board into four distinct cases. One of the following must hold:

Case 1 The board begins with gg

Case 2 The board begins with gr

Case 3 The board begins with r and ends with g

Case 4 The board begins with r and ends with r

In fact, each of these cases represents precisely $\frac{1}{4}$ of the tilings. I claim that cases 2 and 4 cancel each other out in the alternating sum. Note that a g at the beginning of a tiling adds no weight to the tiling, nor does an r at the end. Thus, for each tiling in case 2, we can change the leading g to a trailing r and obtain a tiling in case 4 with equal weight but opposite parity of the number of green tiles.

Thus,

$$\sum_{j=0}^{n+2} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = \sum_{\substack{j=0, \\ \text{tiling begins with gg}}}^{n+2} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} + \sum_{\substack{j=0, \\ \text{tiling begins with r} \\ \text{and ends with g}}}^{n+2} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}$$

In the case of the tilings beginning with gg, note that adding gg to the beginning of a tiling of length n changes neither its weight nor its parity, so

$$\sum_{\substack{j=0, \\ \text{tiling begins with gg}}}^{n+2} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}$$

The remaining case counts tilings of length $n+2$ that begin with r and end with g. We can construct these from tilings of length n by first adding a leading r and then a trailing g. Assume we have a tiling T of length n with j green squares and $n-j$ red squares. Then rT will have weight $q^j q^{w_T}$ since the leading r increases the weight of each green square by 1. Adding a trailing g, we see that the final g has weight $n-j+1$ (the number of red squares before it), so the tiling rTg has weight $q^j q^{w_T} q^{n-j+1} = q^{w_T} q^{n+1}$. Furthermore, adding one red and one green square has changed the parity of the number of green squares, so

$$\sum_{\substack{j=0, \\ \text{tiling begins with r} \\ \text{and ends with g}}}^{n+2} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = \sum_{j=0}^{n} q^{n+1} (-1)^{j+1} \begin{bmatrix} n \\ j \end{bmatrix}$$

$$= -q^{n+1} \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}$$

and therefore

$$\sum_{j=0}^{n+2} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = (1 - q^{n+1}) \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix}$$

as desired.

Thus, since

$$\sum_{j=0}^{2} (-1)^j \begin{bmatrix} 2 \\ j \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1 - (1+q) + 1 = (1-q)$$

by induction, we have

$$\sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = (1-q)(1-q^3)(1-q^5)\dots(1-q^{n-1})$$

for even n.

This inductive proof suggests a method for constructing an almost-bijection. Given a tiling T of an $(n+2)-$board, if it falls under case 2 or 4, pair it up with the corresponding tiling of case 4 or 2. Otherwise, it must have either begun with gg or begun with r and ended with g. Now strip off these two tiles to get a tiling of length n. If this new tiling is of type 2 or 4, change it to type 4 or 2 to get a different tiling of the same weight but opposite parity. Then, add back the leading gg or the leading r and trailing g to get a new tiling T' of length $n+2$. T' will always have the same weight as T because adding a leading gg never adds weight to a tiling, whereas the added weight from a leading r and a trailing g depends only upon the *length* of the interior. Since T' and T are constructed by adding $gg-$ or $r-$

$-g$ to tilings of the same weight, T' and T have the same weight. Note also that T and T' still retain opposite parity. If, however, our tiling of length n is of type 1 or 3, we repeat the process from before, namely:

- Strip off the $gg-$ or $r- \ -g$ and check what case the new tiling falls under.
- If it's case 2 or 4, switch it to case 4 or 2 then sequentially add back the tiles you stripped off to get a tiling T' of length $n+2$ with the same weight as T
- If it's case 1 or 3, repeat these steps again

This gives us a bijection in almost all instances. The only types of tilings we're missing are those for which this procedure eventually gives us the empty 0-board. Such a tiling can be built up by starting with the 0-board and repeatedly adding either $gg-$ prefixes or an $r-$ prefix and a $-g$ suffix. Note that adding a $gg-$ does nothing to the weight or the parity, whereas adding an $r- \ -g$ in the ith step changes the parity and adds weight $q^{2(i-1)+1}$. Thus, the weights of the tilings this bijection misses are: $(1-q)(1-q^3)(1-q^5)\dots(1-q^{n-1})$, as claimed.

3.3 Counting Palindromic Tilings

One question we might naturally be interested in is whether palindromic tilings have any special properties. We can consider two different types of palindromic tilings. First we will examine tilings which are the same when reversed, and then we will consider tilings which are the same when we toggle all the tiles from red to green and then reverse the order. We will call these palindromic tilings of the first and second type, respectively. See Figure 3.1 for some sample palindromic tilings of length 6.

Suppose we have a palindromic tiling T of the first type with even length $2n$. Since the tiling is a palindrome of the first type, the colors of the tiles in position i and $(2n-i+1)$ are the same. Further, since the parity of i and $(2n-i+1)$ are opposite, this shows that our tiling must have an even number of green squares–call this number $2k$. We now define the function $P_q(2n,2k)$ as the sum of the weights of the palindromic tilings of length $2n$ using $2k$ green squares.

We can now determine a recurrence relation for $P_q(2n,2k)$.

Question: What is the sum of the weights of the palindromic tilings of the first type of length $2n$ with $2k$ green squares?

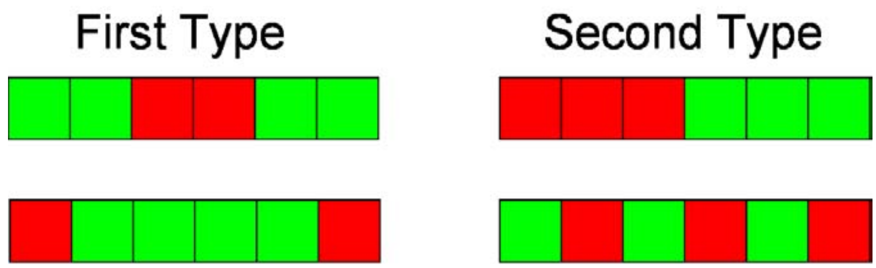


Figure 3.1: Some example palindromic tilings of the first and second type.

Answer 1: By definition, $P_q(2n, 2k)$

Answer 2: Consider the color of the first square. Say the first square is green (and therefore the last square is green as well). Then we have a tiling of the form $gT'g$, where T' is a palindromic tiling of length $2n-2$ with $2k-2$ green squares. (Thus, the sum of the weights of the tilings which can make up T' is $P_q(2n-2, 2k-2)$.) Recall that a leading g adds no weight to a tiling, so the weight of $gT'g$ is just the weight of T' times the weight of the trailing g. Since there are $2(n-k)$ red squares in T', the weight of the trailing green is $q^{2(n-k)}$. Thus, the sum of the weights of the tilings which begin with a green square is $P_q(2n-2, 2k-2)q^{2(n-k)}$.

We can apply a similar process when the first tile is red to see that the sum of the weights of the tilings which begin with a red square is $P_q(2n-2, 2k)q^{2k}$.

Thus, we have the recurrence

$$P_q(2n, 2k) = P_q(2n-2, 2k-2)q^{2(n-k)} + P_q(2n-2, 2k)q^{2k}$$

and the solution to this recurrence is

$$P_q(2n, 2k) = \binom{n}{k} q^{2k(n-k)} \tag{3.5}$$

We now provide a direct combinatorial proof for Equation (3.5).

Question: How many palindromic tilings of the first type of length $2n$ using $2k$ green squares?

Answer 1: By definition, $P_q(2n, 2k)$.

Answer 2: We can think of our tiling as having k pairs of green tiles. Each pair has a green tile in position i and a corresponding tile in position $(2n -$

$i+1$).

I claim that each *pair* of red tiles contributes a weight of q^2 to each *pair* of green tiles. Why? If we consider just a pair of green squares and a pair of red squares and ignore everything else in the tiling, we see that our greens and reds must be arranged as either *grrg* or *rggr*. (The intermediate tiles aren't important since we want to know only the weight contributed to this particular pair of green tiles by this particular pair of red tiles.) If the arrangement is *grrg*, then the red tiles each contribute a weight of q to the second green tile for a total contribution of q^2. On the other hand, if the arrangement is *rggr*, then the initial red contributes a weight of q to each of the green tiles for a total contribution of q^2. Thus, since there are $n-k$ pairs of red squares, each pair of green squares has a weight of $(q^2)^{n-k}$.

Since there are a total of k pairs of green squares, the weight of each tiling is $q^{2(n-k)k}$. Moreover, the number of palindromic tilings of this type is $\binom{n}{k}$ since the palindromic tiling is completely determined by tiling the first n squares with k greens. Hence, the total weight of all palindromic tilings of length $2n$ with $2k$ green squares is

$$\binom{n}{k} q^{2(n-k)k}.$$

Now consider the case of odd palindromic tilings. I claim

$$P_q(2n+1, 2k) = q^k P_q(2n, 2k) = \binom{n}{k} q^{2k(n-k)} q^k \tag{3.6}$$

and

$$P_q(2n+1, 2k+1) = q^{n-k} P_q(2n, 2k) = \binom{n}{k} q^{2k(n-k)} q^{n-k}. \tag{3.7}$$

Proof of 3.6: $P_q(2n+1, 2k)$ counts the number of palindromic tilings of the first kind with $2k$ green squares and $2n-2k+1$ red squares. Since there are an odd number of red squares, the center square must be red. Hence, we can think of this palindromic tiling as a palindromic tiling of length $2n$ with $2k$ green squares and a red square inserted in the middle. Adding a red square in the middle adds weight q to each green square after the middle, so it adds a total weight of q^k to the tiling. Hence,

$$P_q(2n+1, 2k) = q^k P_q(2n, 2k) = \binom{n}{k} q^{2k(n-k)} q^k$$

as desired.

Proof of 3.7: Proceed similarly to the previous proof. $P_q(2n+1, 2k+1)$ counts the number of palindromic tilings of the first kind with $2k+1$ green squares and $2n-2k+1$ red squares. Since there are an odd number of green squares, the center square must be green. Hence, we can think of this palindromic tiling as a palindromic tiling of length $2n$ with $2k$ green squares and a green square inserted in the middle. The green square in the middle will have weight q^{n-k}, so it contributes a weight of q^{n-k} to the tiling. Hence,

$$P_q(2n+1, 2k) = q^{n-k} P_q(2n, 2k) = \binom{n}{k} q^{2k(n-k)} q^{n-k}$$

as desired.

Finally, let $P'_q(2n)$ be the sum of the weights of the palindromic tilings of the *second* type with length $2n$. (Clearly there are no such tilings of length $2n+1$ since tile $n+1$ would have to be both green and red. Also, we need not specify the number of green tiles since exactly half of the tiles in a palindromic tiling of the second type must be green.) I claim that:

$$P'_q(2n) = (1+q)(1+q^3)(1+q^5)\cdots(1+q^{2n-1}).$$

Proof: We prove this constructively. Each palindromic tiling of the second type can be created by successive additions of tiles to the beginning and end of a previous palindromic tiling. To create a palindromic tiling of length $2n$, start with the empty board and successively add either a leading g and a trailing r or a leading r and a trailing g a total of n times. As we saw in Section 3.2, adding a leading g and a trailing r does nothing to the weight of the tiling. On the other hand, adding a leading r and a trailing g to a tiling of length $2i$ contributes weight q^{2i+1}. Hence, the first addition gives the tiling a weight of 1 or q. The second addition contributes weight 1 or q^3. Continue in this fashion until the last addition, which contributes a weight of 1 or q^{2n-1} That is, the total weight of the palindromic tilings of the second type of length $2n$ is $(1+q)(1+q^3)(1+q^5)\ldots(1+q^{2n-1})$, as claimed.

3.4 q-Multichoose and the q-Binomial Series

Recall the combinatorial definition of "multichoose". In the tiling interpretation of the standard binomial coefficients, we can define $\left(\binom{n}{k}\right)$ as the number of tilings of a board of length n using $n-1$ red tiles and k green tiles which are stackable according to the following rules:

- Number the locations on the board from 0 through $n-1$.
- The locations of the red tiles are all predetermined. We place one red tile at the bottom of the stacks at locations 1 through $n-1$.
- We now place the remaining k green tiles one at a time. Each green tile may be placed at the top of any existing stack in locations 0 through $n-1$. There is no limit on how many tiles are allowed in a stack.

Note in particular that the stacks at locations 1 through $n-1$ will necessarily all have at least one tile, since we started by placing red tiles there. The stack at location 0, on the other hand, has no red tiles and could possibly end up with no tiles at all.

One way to interpret $\left(\binom{n}{k}\right)$ is the number of ways to distribute k identical candies to n students. Our interpretation with stackable tiles is equivalent to the student/candy interpretation if we think of the number of green tiles in stack i as the number of candies given to student i. It is also easy to see that

$$\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$$

We simply map a stacked tiling on the left side of the equation to an unstacked tiling on the right by toppling the stacks. That is, a red with three greens on it becomes the sequence *rggg*.

If we now give a method for calculating the weight of a stacked tiling, then we will have a q-analog of multichoose. The method we choose is that each red square gets weight 1 and each green square stacked on position i gets weight q^i. See Figure 3.2 for an example in which we map a stackable tiling on positions 0 through 3 to a tiling of length 7. A similar interpretation, phrased as the selection of balls from boxes rather than the selection of locations for stackable tiles, is provided by John Konvalina in [9].

The following identity, called the q-binomial series, is taken from Andrews and Eriksson's *Integer Partitions* [2], where it is proved using the

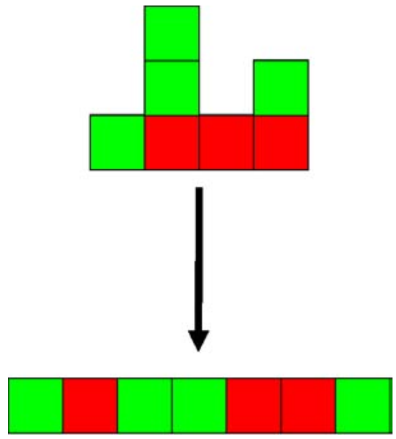


Figure 3.2: An example stackable tiling and the corresponding unstacked tiling, both of weight q^5. To create the unstacked tiling from the stacked tiling, we simply topple each stack.

partition-in-a-box interpretation. Here, we prove it with the stackable tiling interpretation of q-multichoose.

$$\prod_{j=1}^{n} \frac{1}{1 - zq^j} = \sum_{m=0}^{\infty} q^m \left[\left[\begin{matrix} n \\ m \end{matrix}\right]\right] z^m \tag{3.8}$$

Rather than prove this directly, we will first prove the following modified identity:

$$\prod_{j=0}^{n-1} \frac{1}{1 - zq^j} = \sum_{m=0}^{\infty} \left[\left[\begin{matrix} n \\ m \end{matrix}\right]\right] z^m \tag{3.9}$$

Proof of 3.9:
Question: Give a generating function in z for the weighted number of stackable tilings on positions 0 through $n-1$. (The power of z counts the total number green tiles used.)
Answer 1: We have infinitely many choices for the number of green tiles on each position. At position 0, we may choose to have either 0 or 1 or 2 or … green tiles, so our generating function has a factor of $(1 + z + z^2 + z^3 + \ldots)$. Next, at position 1, we may choose to have either 0 or 1 or 2 or … green tiles, so our generating function has a factor of $(1 + zq + (zq)^2 + (zq)^3 + \ldots)$. Now we choose the number of green tiles on position 2. Since each green

tile gets a weight of q^2, our generating function has a factor of $(1 + zq^2 + (zq^2)^2 + \ldots)$. Continuing this pattern up to position $n-1$, our overall generating function is

$$(1 + z + z^2 + \ldots) \ldots (1 + zq^{n-1} + (zq^{n-1})^2 + \ldots) = \prod_{j=0}^{n-1} \frac{1}{1 - zq^j}$$

Answer 2: By definition, the sum of the weights of the stackable tilings on positions 0 through $n-1$ using m green squares is $\left[\begin{bmatrix} n \\ m \end{bmatrix}\right]$. In other words, our generating function is

$$\sum_{m=0}^{\infty} \left[\begin{bmatrix} n \\ m \end{bmatrix}\right] z^m$$

Therefore, since both are answers to the same combinatorial question,

$$\prod_{j=0}^{n-1} \frac{1}{1 - zq^j} = \sum_{m=0}^{\infty} \left[\begin{bmatrix} n \\ m \end{bmatrix}\right] z^m$$

as desired.

Now to obtain Equation (3.8) from Equation (3.9), simply replace z in (3.9) with the quantity (qz). Combinatorially, this substitution is equivalent to adding the restriction that our generating function should give the weights of the tilings on positions 0 through n where tiles may not be placed on position 0. That is, no green tiles may be placed before the first red tile.

3.5 Analogue to the Sums of Consecutive Integers

The binomial identity

$$\sum_{k=1}^{n} k = \binom{n+1}{2} \tag{3.10}$$

can be proved by considering the location of the last green square in a tiling of length $n+1$ with 2 green squares. This binomial identity and its counterparts for squares and cubes in the following two sections were taken from Benjamin and Quinn's *Proofs that Really Count* [3].

One q-analogue of this theorem is

$$\sum_{k=1}^{n} [k]_q q^{k-1} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} \tag{3.11}$$

and the proof is analogous to the non q-ified case.

Question: What is the sum of the weights of the tilings of length $n+1$ using 2 green squares?

Answer 1: By definition, $\begin{bmatrix} n+1 \\ 2 \end{bmatrix}$

Answer 2: Consider the location of the second green square. Label the locations of the board from 1 to $n+1$ and call the position of the second green square $k+1$. Then the second green square has $k-1$ red squares before it and therefore has weight q^{k-1}. The first green square, on the other hand, can be preceded by 0 or 1 or 2 or ... or $k-1$ red squares. Thus, the total weight of the tilings with the second green square in position $k+1$ is $q^{k-1}(1+q+\ldots+q^{k-1}) = q^{k-1}[k]_q$. So the answer to our question is the sum over all possible locations $k+1$ of $q^{k-1}[k]_q$. That is,

$$\sum_{k+1=2}^{n+1} q^{k-1}[k]_q = \sum_{k=1}^{n} q^{k-1}[k]_q,$$

as desired.

Michael Schlosser's paper "q-Analogues of the sums of consecutive integers, squares, cubes, quarts, and quints" [11] gives a different identity for the sum of q-integers:

$$\sum_{k=1}^{n} [k]_q q^{2(n-k)} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} \tag{3.12}$$

For this version of the theorem, we simply let k count the total number of squares to the right of the first green square.

3.6 Analogue to the Sums of Integer Cubes

From the identity on the sum of integers, we will proceed to the corresponding identity giving the sum of integer cubes. The identity on the sum of integer squares turns out to be slightly more complicated and will be handled in the next section.

Garrett and Hummel [7] give the following nice identity for the sum of q-cubes.

$$\sum_{k=1}^{n} q^{k-1}\left(\frac{1-q^k}{1-q}\right)^2\left(\frac{1-q^{k-1}}{1-q^2}+\frac{1-q^{k+1}}{1-q^2}\right)=\begin{bmatrix} n+1 \\ 2 \end{bmatrix}^2 \tag{3.13}$$

Using their formula as inspiration, I discovered the modified version below.

Theorem:

$$\sum_{k=1}^{n} q^{k-1}[k]_q^3=\begin{bmatrix} n+1 \\ 2 \end{bmatrix}^2+(q-1)\sum_{k=1}^{n} q^{k-1}[k]_q\begin{bmatrix} k \\ 2 \end{bmatrix} \tag{3.14}$$

Note that as we let q go to 1 in (3.14), the final term drops out and we obtain the binomial identity for the sum of cubes:

$$\sum_{k=1}^{n} k^3=\binom{n+1}{2}^2 \tag{3.15}$$

We provide a combinatorial proof of the slightly rearranged identity:

$$\sum_{k=1}^{n} q^{k-1}[k]_q^3+\sum_{k=1}^{n} q^{k-1}[k]_q\begin{bmatrix} k \\ 2 \end{bmatrix}=\begin{bmatrix} n+1 \\ 2 \end{bmatrix}^2+q\sum_{k=1}^{n} q^{k-1}[k]_q\begin{bmatrix} k \\ 2 \end{bmatrix} \tag{3.16}$$

Before doing so, it will be helpful to introduce some new notation. Let $(x_1, x_2, \ldots, x_a)_n$ denote the weighted tiling of a board of length n with green squares at positions $x_1, x_2, \ldots, x_a$ and red squares elsewhere. We require $1 \leq x_1 < x_2 < \cdots < x_a \leq n$. We can calculate the weight of this tiling as

$$q^{\sum_{i=1}^{a} x_i - i}.$$

For example, $(a)_n$ would denote a length n tiling of weight q^{a-1} with a single green square on cell a. See Figure 3.3 for some illustrations of this new notation.

Now for the four terms in Equation (3.16), we define four sets for which they give the weight.

Let

$$\begin{aligned}
\mathcal{A} &= \{(a)_{n+1}, (b)_{n+1}, (c,d)_{n+1} \mid a, b < d\} \\
\mathcal{B} &= \{(e,f)_{n+1}, (g,h)_{n+1} \mid f < h\} \\
\mathcal{C} &= \{(i,j)_{n+1}, (l,m)_{n+1}\} \\
\mathcal{D} &= \{(r)_{n+1}, (s)_{n+1}, (t,u)_{n+1} \mid r < s < u\}
\end{aligned}$$

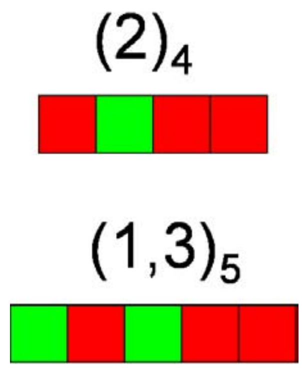


Figure 3.3: Two example tilings along with their specifications under the notation just introduced.

If we let $|\mathcal{A}|$ represent the sum of the weights of the tilings in $\mathcal{A}$, then by letting $k = d - 1$,

$$|\mathcal{A}| = \sum_{k=1}^{n} q^{k-1}[k]_q^3$$

Similarly, by letting $k = u - 1$, the sum of the weights of the tilings in $\mathcal{D}$ is given by

$$|\mathcal{D}| = q \sum_{k=1}^{n} q^{k-1}[k]_q \begin{bmatrix} k \\ 2 \end{bmatrix}$$

By letting $k = h - 1$, the sum of the weights of the tilings in $\mathcal{B}$ is given by

$$|\mathcal{B}| = \sum_{k=1}^{n} q^{k-1}[k]_q \begin{bmatrix} k \\ 2 \end{bmatrix}$$

Finally, the sum of the weights of the tilings in $\mathcal{C}$ is given by

$$|\mathcal{C}| = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}^2$$

Now to prove the identity, we want a weight-preserving bijection from $\mathcal{A} \cup \mathcal{B}$ to $\mathcal{C} \cup \mathcal{D}$. Note that $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{D} \subset \mathcal{A}$ so most of our bijection can be accomplished with the identity map.

- For each element of $\mathcal{B}$, map via the identity map into $\mathcal{C}$. All elements of $\mathcal{C}$ are hit except for those where $j \geq m$.

- For each element of $\mathcal{A}$ with $a < b$, map via the identity map into $\mathcal{D}$. (This covers all of $\mathcal{D}$.)
- For the remaining elements of $\mathcal{A}$ with $a \geq b$, take $(a)_{n+1}, (b)_{n+1}, (c,d)_{n+1}$ to $(c,d)_{n+1}, (b, a+1)_{n+1}$, which is an element of $\mathcal{C}$ with $j \geq m$.

This map is 1-1, onto, and weight preserving, so we have our identity.

3.7 Analogues to the Sum of Integer Squares

Michael Schlosser [11] lists two different q-analogues for the sum of squares:

$$\sum_{k=1}^{n} \frac{(1-q^{2k})(1-q^k)}{(1-q^2)(1-q)} q^{\frac{3}{2}(n-k)} = \frac{(1-q^n)(1-q^{n+1})(1-q^{n+1/2})}{(1-q)(1-q^2)(1-q^{3/2})} \tag{3.17}$$

and

$$\sum_{k=1}^{n} \frac{(1-q^{3k})(1-q^k)}{(1-q^3)(1-q)} q^{2(n-k)} = \frac{(1-q^n)(1-q^{n+1})(1-q^{2n+1})}{(1-q)(1-q^2)(1-q^3)} \tag{3.18}$$

Here I present two further q-analogues of my own construction.

Theorem:

$$\sum_{k=1}^{n} q^{k-1}[k]_q^2 = 2q^2 \begin{bmatrix} n+1 \\ 3 \end{bmatrix} + \sum_{k=1}^{n} q^{k-1}[k]_{q^2} \tag{3.19}$$

Proof: Question: What is the sum of the weights of the tilings in the set

$$\{(a)_{n+1}, (b,c)_{n+1} | a < c\}$$

Answer 1: Consider the value of c. If $c = k+1$, then the green tile it represents has weight $q^{k+1-2} = q^{k-1}$. Once we've picked c, we can independently choose a and b to be anything from 1 to k. These choices give a total weight of $[k]_q \times [k]_q$. Hence, the sum of the weights of all tilings in the set is

$$\sum_{k=1}^{n} q^{k-1}[k]_q^2$$

Answer 2: Split the problem up into cases based on whether or not $a = b$. If $a < b$, then we can create $(a)_{n+1}, (b,c)_{n+1}$ by taking the tiling $(a,b,c)_{n+1}$ and breaking a out into a separate board. Thus, the number of such tilings is $\binom{n+1}{3}$ and the weight of $(a)_{n+1}, (b,c)_{n+1}$ is greater than the corresponding tiling $(a, b, c)_{n+1}$ by a factor of q^2. Thus, the sum of the weights of the tilings with $a < b$ is

$$q^2 \begin{bmatrix} n+1 \\ 3 \end{bmatrix}$$

A similar argument works in the case where $a > b$. In this case, we also have the sum of the weights of the tilings is

$$q^2 \begin{bmatrix} n+1 \\ 3 \end{bmatrix}$$

Finally, we have to consider the case where $a = b$. As in our first solution, we consider the value of c. If $c = k+1$ then that tile gets weight q^{k-1}. As in Answer 1, we are left to choose a and b, but now we have the additional restraint that $a = b$. If we put them in position 1, then they have a combined weight of 1. In position 2, they have combined weight q^2. In position i they have weight q^{2i-2}. Thus, the weights of the tilings with $c = k+1$ are given by

$$q^{k-1}(1 + q^2 + q^4 + \ldots + q^{2k-2}) = q^{k-1}[k]_{q^2}$$

Putting these cases together, the sum of the weights of the tilings in the set is

$$2q^2 \begin{bmatrix} n+1 \\ 3 \end{bmatrix} + \sum_{k=1}^{n} q^{k-1}[k]_{q^2}$$

as desired.

For our second identity, we start with a different formulation for the sum of integer squares. Duane DeTemple [5] has provided a combinatorial proof for the following identity by counting flagpole arrangements.

$$\sum_{k=1}^{n-1} k^2 = \frac{1}{4}\binom{2n}{3} \tag{3.20}$$

I modified this identity to a q-analogue of the following form:

$$(1 + 2q + q^2) \sum_{k=1}^{n-1} [k]_{q^2}^2 q^{2k-1} = \begin{bmatrix} 2n \\ 3 \end{bmatrix} + (q-1) \sum_{k=1}^{n-1} [2k-1]_q [2k]_q q^{2k-2} \tag{3.21}$$

As in the case of the sum of cubes, note that if we let $q = 1$, the whole final term drops out and leaves our binomial identity (3.20).
I will provide a combinatorial proof for the slightly rearranged identity:

$$(1+2q+q^2)\sum_{k=1}^{n-1}[k]_{q^2}^2 q^{2k-1} + \sum_{k=1}^{n-1}[2k-1]_q[2k]_q q^{2k-2} = \begin{bmatrix}2n\\3\end{bmatrix} + q\sum_{k=1}^{n-1}[2k-1]_q[2k]_q q^{2k-2} \qquad (3.22)$$

Proof of 3.22: Our proof will consist of two steps. First, we will define a set whose weight is given by each of the four terms in 3.22. Secondly, we will give a weight-preserving bijection between the sets on the left-hand side and right-hand side.

Using the same notation as in Section 3.6, let

$$\begin{aligned}
\mathcal{A} &= \{(a)_{2n},(b,c)_{2n} \mid a<c, c \text{ odd}\}\\
\mathcal{B} &= \{(e,f,g)_{2n}\}\\
\mathcal{C} &= \{(r)_{2n},(s,t)_{2n} \mid t \le 2n-2, r\le t, t \text{ even}\}\\
\mathcal{D} &= \{(u)_{2n},(v,w)_{2n} \mid u<w-1, w \text{ odd}\}
\end{aligned}$$

Note that

$$\sum_{k=1}^{n-1}[k]_{q^2}^2 q^{2k-1}$$

gives the weight of the tilings in the set $\{(a)_{2n},(b,c)_{2n} | a,b<c, \text{with } a,b,c \text{ odd}\}$. Multiplying by the factor $(1+2q+q^2)$ allows us to remove the restriction that a and b be odd. (Why? Multiplying by 1 leaves both a and b odd. Multiplying by $2q$ either leaves a odd and adds one to b or leaves b odd and adds one to a. Multiplying by q^2 adds one to both a and b, making them both even.) Thus,

$$|\mathcal{A}| = (1+2q+q^2)\sum_{k=1}^{n-1}[k]_{q^2}^2 q^{2k-1}.$$

By our definition of the q-binomial coefficient, the sum of the weights of the tilings in $\mathcal{B}$ is given by

$$|\mathcal{B}| = \begin{bmatrix}2n\\3\end{bmatrix}$$

By letting $t = 2k$, the sum of the weights of the tilings in $\mathcal{C}$ is given by

$$|\mathcal{C}| = \sum_{k=1}^{n-1} [2k-1]_q [2k]_q q^{2k-2}$$

Finally, by letting $w = 2k+1$ the sum of the weights of the tilings in of $\mathcal{D}$ is given by

$$|\mathcal{D}| = \sum_{k=1}^{n-1} [2k-1]_q [2k]_q q^{2k-1}$$

Now we must define a weight-preserving bijection from $\mathcal{A} \cup \mathcal{C}$ to $\mathcal{B} \cup \mathcal{D}$.

Since $\mathcal{D} \subset \mathcal{A}$, it makes sense to rewrite $\mathcal{A}$ as $\mathcal{E} \cup \mathcal{F}$ where $\mathcal{E} = \mathcal{D}$ and $\mathcal{F}$ is $\mathcal{A}$ with the additional restriction that $a = c - 1$.

The weight-preserving bijection we use is:

$$\begin{array}{rcl} \mathcal{E} & \rightarrow & \mathcal{D} \\ (c-1)_{2n}, (b,c)_{2n} \in \mathcal{F} & \rightarrow & (b,c,c+1)_{2n} \in \mathcal{B} \\ (r)_{2n}, (s,t)_{2n} \in \mathcal{C} & \rightarrow & \begin{cases} (s,r,t+2)_{2n} \in \mathcal{B} \text{ if } r > s \\ (r,r+1,t+1)_{2n} \in \mathcal{B} \text{ if } r = s \\ (r,s+1,t+1)_{2n} \in \mathcal{B} \text{ if } r < s \end{cases} \end{array}$$

It is easy to check that this bijection is 1-1, onto, and weight-preserving, and therefore that it completes our proof of the identity.

3.8 The q-Lucas' Theorem

Lucas' theorem allows us to simplify binomial coefficients modulo a prime. If we let p be a prime and let a and b be non-negative integers with $0 \leq a, b < p$, then Lucas' theorem says:

$$\binom{pn+a}{pk+b} \equiv \binom{n}{k}\binom{a}{b} (\text{mod } p) \tag{3.23}$$

Proof sketch:

Question: How many ways (mod p) can we tile a board of dimensions $p \times n$ and a strip of dimensions $a \times 1$ using $pk + b$ green unit squares and $p(n-k) + (a-b)$ red squares?

Answer 1: From the $pn + a$ total locations, select which $pk + b$ hold green squares. There are $\binom{pn+a}{pk+b}$ ways to do this.

Answer 2: Since we're working modulo p, let's start by grouping as many of these tilings as possible into sets of size p.

Start by looking at the first column of the board. If not all the squares in this column are the same color, then moving the top square to the bottom of the column and shifting the rest of the squares up by one position will create a different column. In fact, since p is prime, repeating this process $p-1$ times will create p distinct columns. Thus, whenever the first column is not all the same color, we can place it in a group of p distinct tilings of the board. Since we are only considering the number of tilings modulo p, this means we can ignore any tilings in which the first column is not all the same color.

If the first column is all the same color, move on to the second column and check if it is all one color. If not, perform the same shifting procedure. If it is all the same color, move onto the third column. Continue in this fashion until you reach the right end of the board.

Now the only tilings of the board and the strip which haven't been put into a set of size p are the ones in which each column of the board is just a single color. How many such tilings are there? There are $\binom{n}{k}$ ways to tile the board. (Just choose which columns are green.) Then there are $\binom{a}{b}$ ways to place the remaining b green tiles onto the strip of length a. Hence, our answer is

$$\binom{n}{k}\binom{a}{b},$$

as desired.

The equivalent q-identity is

$$\begin{bmatrix} pn+a \\ pk+b \end{bmatrix} \equiv \binom{n}{k} \begin{bmatrix} a \\ b \end{bmatrix} (\text{mod } [p]_q) \tag{3.24}$$

One of the key observations required for the upcoming proof is that

$$q^p \equiv 1 (\text{mod } [p]_q).$$

This is true because

$$q^p - 1 = [p]_q(q-1).$$

Our proof will also require a lemma on the effects of cycling through a tiling as we did with the columns in the previous proof.

Lemma 3.1. *For a prime p, take a tiling of a strip of length p with at least one red tile and at least one green tile. Following the procedure outlined in the proof of 3.23, create p distinct tilings from this one by successively removing the front tile*

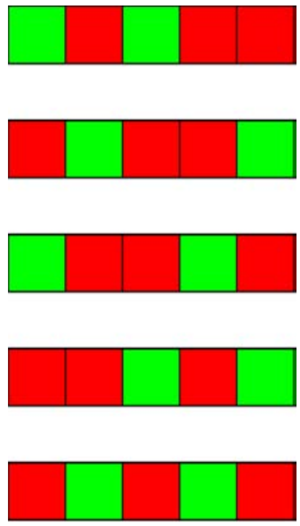


Figure 3.4: The shifting procedure described in Lemma 3.1 applied to a tiling of length 5. Note that this procedure does indeed produce 5 distinct tilings. From top to bottom, their weights are q, q^4, q^2, q^5, and q^3. The sum of these weights is $q + q^2 + q^3 + q^4 + q^5 = q[5]_q$, which is a multiple of $[5]_q$, as desired.

and placing it at the back. The sum of the weights of these p tilings is a multiple of $[p]_q$. *(See Figure 3.4 for an example of this shifting procedure applied to a tiling of length 5.)*

Proof: Note that it is equivalent to show that the exponent on the weight of each of these p tilings is distinct mod p. (Why? If each tiling has a different exponent (mod p), then their sum will be equivalent to $1 + q + q^2 + \cdots + q^{p-1} \pmod{[p]_q}$. Since $1 + q + q^2 + \cdots + q^{p-1} = [p]_q$, this shows that the sum of the weights is congruent to $[p]_q$, as desired.)

Now let's look at the effect on a tiling of moving the front tile to the back. Say our tiling has k green squares and $p - k$ red squares (where $0 < k < p$ since we have both red and green squares.) If the front tile is green, moving it to the back will change the weight of the tiling by a factor of q^{p-k} since we now have an additional green square counting all the reds. If, on the other hand, the front tile is red, moving it to the back will change the weight of the tiling by a factor of q^{-k} since the red tile was previously counted by all the greens and is now counted by none of them. However, since $q^p \equiv 1$, these actions have equivalent effects on the weight of the

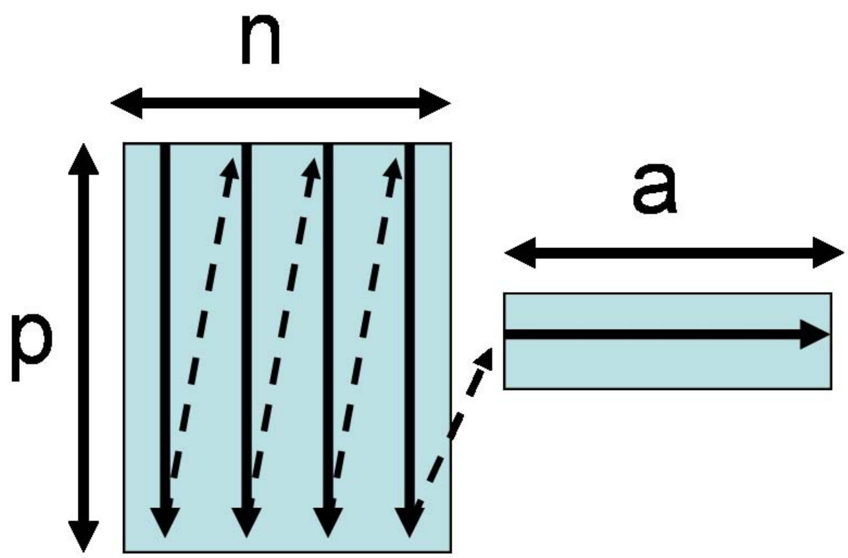


Figure 3.5: The board and strip described in the question for the proof of Equation (3.24). The arrows on the diagram indicate the order in which we will consider tiles when calculating the weight of the board and strip.

tiling (mod $[p]_q$).

Hence, if the weight of the first tiling is q^s, the weight of the second tiling is congruent to q^{s-k}. The weight of the third is congruent to q^{s-2k}, and so forth until the last tiling which has weight $q^{s-(p-1)k}$. However, since p is prime and k is strictly between 0 and p, these exponents are all distinct modulo p, as desired.

With this lemma, we are now ready to prove Equation (3.24).

Proof:

Question: What is the sum of the weights of the tilings (mod $[p]_q$) of a board with dimensions $p \times n$ and a strip with dimensions $a \times 1$ using $pk + b$ green unit squares and $p(n-k) + (a-b)$ red squares? Note that to assign a weight to this board/strip combination, we must provide an order for all $pn + a$ locations. We will say that the first location is the upper left corner of the board. From there we proceed down the first column, then down the second, and so on until the end of the board. Finally, we will proceed across the strip so that the last tile is at the right end of the strip. See Figure 3.5 for an illustration of this setup.

Answer 1: Despite the unusual layout, we can still determine the sum of the weights of the tilings in the usual way. Since there are $pn+a$ locations and $pk+b$ green squares, our answer is just

$$\begin{bmatrix} pn+a \\ pk+b \end{bmatrix}.$$

Answer 2: As in the standard proof, look for the first column whose tiles aren't all the same color and perform the same shifting procedure to create p different tilings. Lemma 3.1 showed that the sum of the weights of these columns will be a multiple of $[p]_q$. Furthermore, since rotating the tiles within a column doesn't change the interactions between that column and the rest of the tiling, the sum of the weights of these p tilings is also a multiple of $[p]_q$. Hence, since we are working mod $[p]_q$, we need only consider the tilings upon which we cannot perform this procedure, i.e. the tilings in which every column is monochromatic.

To calculate the sum of the weights of these tilings, we will split them into two parts. First we consider the weight of the board, then the weight of the strip, and finally the interactions between them.

We know the board must have k columns of green tiles and $n-k$ columns of red tiles. Since each column has height p, we see that every red column contributes weight q^{p^2} for each green column after it. However, since we are working modulo $[p]_q$, we know $q^p \equiv 1$ and therefore $q^{p^2} \equiv 1$ as well. Hence, the weight of the board will always be 1, so the contribution from the board is simply the number of arrangements of the k green and $n-k$ red columns, i.e. $\binom{n}{k}$.

The sum of the weights of the strip is easy to obtain. We have a strip of length a with b green squares, so the sum of the possible weights is $\begin{bmatrix} a \\ b \end{bmatrix}$.

Finally, we must consider the interaction between the strip and the board. Each of the b green squares in the strip counts each of the $p(n-k)$ red squares in the board, so we must multiply our answer of $\binom{n}{k}\begin{bmatrix} a \\ b \end{bmatrix}$ by an additional factor of $q^{bp(n-k)}$. However, modulo $[p]_q$, we have $q^{bp(n-k)} \equiv (q^p)^{b(n-k)} \equiv 1$. Thus, our answer is simply $\binom{n}{k}\begin{bmatrix} a \\ b \end{bmatrix}$, as desired.

3.9 An Identity on Tiling Two Boards

The following identity is given with an algebraic proof in John Riordan's *Combinatorial Identities* [10].

$$\binom{n}{x}\binom{n}{y} = \sum_{j\geq 0}\binom{x}{j}\binom{y}{j}\binom{n+j}{x+y} \tag{3.25}$$

We will prove it here combinatorially and then give a q-analogue.

Proof:

Question: From the set of integers 1 through n, how many ways are there to choose a subset of size x and a subset of size y, where these two subsets may overlap?

Answer 1: First choose the subset of size x, then choose the subset of size y. There are $\binom{n}{x}\binom{n}{y}$ ways to do this.

Answer 2: Let X be the subset of size x and Y be the subset of size y. For each selection of subsets, we can create a sequence of x X's and y Y's by listing, for each number 1 through n, the subset or subsets in which that number appears. For example, if $X = \{1,5\}$ and $Y = \{2,4,6\}$, our sequence would be XYYXY since $1 \in X$, $2 \in Y$, $3 \in$ neither, $4 \in Y$, $5 \in X$, and $6 \in Y$. See Figure 3.6 for a graphical illustration of this example. By convention, we will say that if a number appears in both X and Y, we will list the X before the Y in our sequence. For example, $X = \{3,5\}$ and $Y = \{2,3\}$ would lead to the sequence YXYX. ($2 \in Y$, $3 \in X$ and Y, $5 \in X$.)

Let j count the number of times an "XY" appears in our sequence. If we choose which j of the X's and which j of the Y's take part in the "XY" pairings, this uniquely determines the sequence. Therefore, there are $\binom{x}{j}\binom{y}{j}$ sequences in which "XY" appears exactly j times.

Once we have selected a sequence of X's and Y's in which "XY" appears j times, I claim there are $\binom{n+j}{x+y}$ selections of subsets X and Y which fit this specific sequence. That is, we can map each selection of $x+y$ numbers from the set of numbers 1 through $n+j$ onto a distinct pair of subsets X and Y which fit our sequence of X's and Y's. We use the sequence of X's and Y's to determine the specific transformation we must apply in the following way:

Start by writing a zero under each of the X's and Y's in the sequence. For each appearance of "XY" in the sequence, subtract one from the number under the Y and from all numbers after that Y. For example, the sequence XYYXY would lead to $\{0, -1, -1, -1, -2\}$.

Once we have created this sequence of nonpositive numbers, we add them to the list of $x+y$ elements selected from 1 through $n+j$. Determine

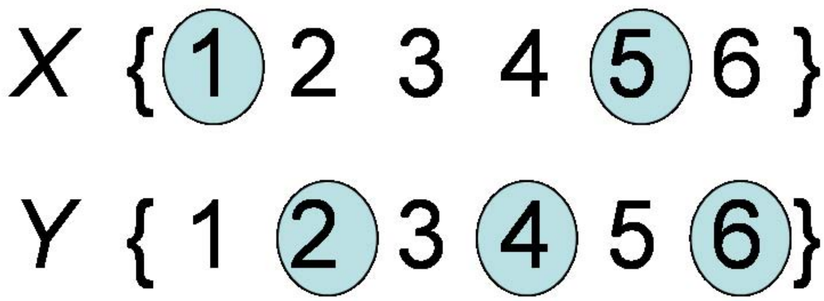


Figure 3.6: A graphical illustration of the example in which $X = \{1, 5\}$ and $Y = \{2, 4, 6\}$. Here we have written out the numbers 1 through n twice and circled those which appear in X on top and those which appear in Y on below. We can then determine the ith element in the sequence of X's and Y's by scanning this diagram from left to right and looking at whether the ith circle appears in the top or bottom row. In this case, we obtain the sequence "XYYXY".

which elements are in the set X and which are in Y in the natural way by matching up this new sequence with the sequence of X's and Y's. This transforms the selection to a list of elements from the numbers 1 through n where repetition is allowed only in locations where an "XY" appears in the sequence.

Thus, our final answer is $\sum\limits_j \binom{x}{j}\binom{y}{j}\binom{n+j}{x+y}$, as desired.

The corresponding q-identity is:

$$\begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ y \end{bmatrix} = \sum_j q^{(x-j)(y-j)} \begin{bmatrix} x \\ j \end{bmatrix} \begin{bmatrix} y \\ j \end{bmatrix} \begin{bmatrix} n+j \\ x+y \end{bmatrix}$$

Proof:

Question: Create a pair of tilings, both of length n, where the first tiling contains x green squares and the second tiling contains y green squares. Let the weight of this pair of tilings be the product of the weights of the individual tilings. What is the sum of the weights of all such pairs of tilings?

Answer 1: The sum of the weights of the first tiling alone is $\begin{bmatrix} n \\ x \end{bmatrix}$ and the sum of the weights of the second tiling alone is $\begin{bmatrix} n \\ y \end{bmatrix}$. Since we calculate the weight of the pair by multiplying the weights of the individual tilings, our answer is simply $\begin{bmatrix} n \\ x \end{bmatrix} \begin{bmatrix} n \\ y \end{bmatrix}$.

Answer 2: Let X be the set consisting of the locations selected for the green tiles in the first tiling and Y be the analogous set for the second tiling. As before, for each selection of X and Y, we can create a sequence of x X's and y Y's by listing, for each number 1 through n, the set or sets in which that number appears. We follow the same convention as before that if a number appears in both X and Y, we will list the X before the Y in our sequence.

Again we let j count the number of times an "XY" appears in our sequence. Once we have selected a sequence of X's and Y's in which "XY" appears j times, as we saw previously, there are $\binom{n+j}{x+y}$ selections of sets X and Y which fit this specific sequence. That is, we can map each selection of $x + y$ green tiles from the positions 1 through $n + j$ onto a distinct pair of sets X and Y which fit our sequence of X's and Y's. For this proof, we will start with this set of single tilings whose weights sum to $\begin{bmatrix} n+j \\ x+y \end{bmatrix}$ and examine how the transformation to pairs of tilings affects the weight.

We will perform the transformation in two steps. The first step is to break out *only* the Y tiles which are involved in "XY" pairs into a second board and perform the required left-shifting transformation on all tiles as described in the standard binomial proof. For example, with the sequence $\underline{XY}Y\underline{XY}X$ and the selection of green tiles on locations $\{1, 2, 3, 5, 8, 13\}$, we would create the pair of tilings with green tiles on locations $\{1, 2, 4, 11\}$ and $\{1, 6\}$, as illustrated in Figure 3.7. It is important to remember that the green tile on position 2 in the top tiling really belongs on the same position in the bottom tiling, but we will handle that with the second step of our transformation.

Now we must handle the question of how this first step of the transformation affected the weight of the tilings. Notice that the first Y broken out gains weight equal to q raised to the number of tiles before it which aren't involved in "XY" pairs. The removal of that Y tile from the first board is negated by the fact that all subsequent tiles are shifted one position to the left. In the previous example, the first Y broken out initially had weight q^0 and continues to have weight q^0 after the first step of the transformation, as anticipated. From there, we proceed to look at all the subsequent Y tiles broken out. Each gains weight given by q to the number of tiles before it which aren't involved in "XY" pairs. In the example above, the second Y tile removed from the top board initially had weight q^3 and ends up with

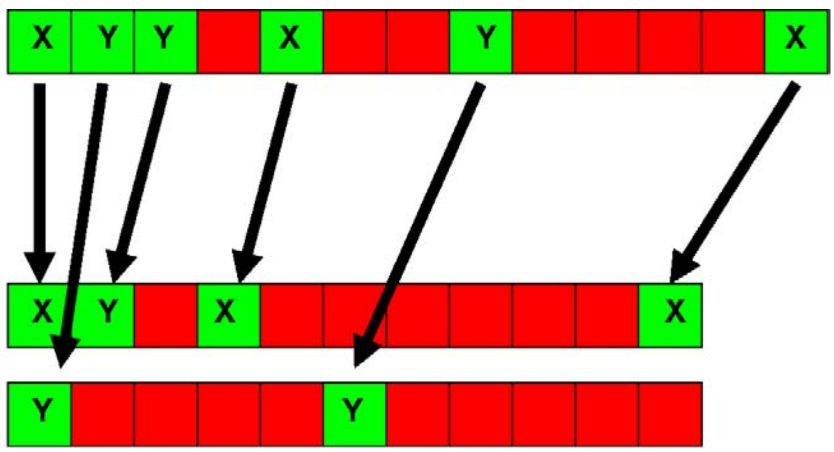


Figure 3.7: The first step of our transition for the tiling with tiles on locations $\{1, 2, 3, 5, 8, 13\}$ and string "XYYXYX". Note that the only Y tiles which have been moved down to the lower board were those which were part of an "XY" pair. This leaves a Y tile on position 2 in the top board. We have decided how far left to move each tile based on the transformation described in the proof of the standard binomial equation.

weight q^4 after this step of the transformation, reflecting the single Y before it not involved in an "XY" pairing.

Hence, in total, this first step of the transformation increases the exponent on the weight by adding to it the sum of the number of unpaired tiles before each Y involved in an "XY" pair. This value is entirely dependent upon which of the X's and Y's were selected to be involved in pairing. In fact, over all sequences with j "XY" pairs, the sum contributed by unpaired X's is $\begin{bmatrix} x \\ j \end{bmatrix}$ and the sum contributed by unpaired Y's is $\begin{bmatrix} y \\ j \end{bmatrix}$.

Now we perform the second step of our transformation. All that is left to do is move the Y's which weren't involved in "XY" pairs from the first board to the second board (without changing their position.) First we consider the effect of this change on the tilings involved in "XY" pairs. Note that if moving a tile down to the second board increases the weight of an X in an "XY" pair it must simultaneously decrease the weight of the corresponding "Y" in the second board. Hence, the net effect on the weight of tiles involved in "XY" pairings is zero. All that remains is to look at the weight contributed to unpaired X's and Y's by this change. We should get an additional weight of q for each interaction between an unpaired X and an unpaired Y. If the X comes first, then the Y will count that one extra red tile when moved down to the second board. On the other hand, if the Y comes first, then it empties an extra tile to be counted by the X when it moves to the second board. Thus, the total effect on the weight of the tiling is to add to its exponent the number of unpaired X's multiplied by the number of unpaired Y's. That is, this second step multiplies the weight of the tiling by a factor of $q^{(x-j)(y-j)}$.

With the transformation complete, we can now assess the sum of the weights of the tilings for a fixed j. We find this sum to be

$$\begin{bmatrix} x \\ j \end{bmatrix} \begin{bmatrix} y \\ j \end{bmatrix} \begin{bmatrix} n+j \\ x+y \end{bmatrix} q^{(x-j)(y-j)}$$

as desired, and thus our final answer to the question of the sum of the weights of the pairs of tilings is

$$\sum_j q^{(x-j)(y-j)} \begin{bmatrix} x \\ j \end{bmatrix} \begin{bmatrix} y \\ j \end{bmatrix} \begin{bmatrix} n+j \\ x+y \end{bmatrix},$$

completing the proof.

Chapter 4

Identities Requiring more than two Colors

4.1 A Three-Color Identity

The following binomial identity, despite its simple proof and the simple form of its generalization, nevertheless requires that we introduce a third color tile to provide a tiling proof.

$$\binom{n}{m}\binom{m}{p} = \binom{n}{p}\binom{n-p}{m-p} \tag{4.1}$$

Proof sketch:

Question: How many ways are there to choose a committee of size m with a subcommittee of size p from a class of n students?

Answer 1: $\binom{n}{m}\binom{m}{p}$. First choose the committee, then choose the subcommittee.

Answer 2: $\binom{n}{p}\binom{n-p}{m-p}$. First choose the subcommittee, then choose the remainder of the committee.

We see that to translate this into a proof using tilings, we will need to have a way to distinguish between tiles which are "in the subcommittee", "in the committee", and neither. We will choose to do so by introducing a third color of tiles. Then our question for the standard binomial identity becomes: "How many ways can we tile a board of length n using $n-m$ red squares, $m-p$ green squares, and p super-green squares?"

To q-ify this interpretation, we need a way of calculating the weight of a tiling with red, green, and super-green squares. As before, each red square gets weight 1 and each green square gets weight q^r where r is the number of red squares before it. However, each super-green square gets weight q^s where s is sum of the number of red squares and green squares before it.

Under this interpretation, we get the q-identity

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ p \end{bmatrix} = \begin{bmatrix} n \\ p \end{bmatrix} \begin{bmatrix} n-p \\ m-p \end{bmatrix}. \tag{4.2}$$

Proof:

Question: What is the sum of the weights of the tilings of a board of length n using $n-m$ red squares, $m-p$ green squares, and p super-green squares?

Answer 1: First choose the locations of the m green and super-green squares. Each red square contributes weight q to each of these squares, so we have a factor of $\begin{bmatrix} n \\ m \end{bmatrix}$. Next, choose which p of these m squares are super-green. We've already accounted for the weight contributed to the super-green squares from the red squares, so multiplying by $\begin{bmatrix} m \\ p \end{bmatrix}$ gives us the weight of the super-green squares by including the contribution of the green squares. Thus, our final answer is

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ p \end{bmatrix}.$$

Answer 2: First choose the location of the p super-green squares. Since they get weight q for each non-super-green square before them, the total weight of the super-green squares is $\begin{bmatrix} n \\ p \end{bmatrix}$. Next, ignore the super-green squares and choose where to place the $m-p$ green squares in the remaining $n-p$ spots. This gives us a total weight of $\begin{bmatrix} n-p \\ m-p \end{bmatrix}$ for the green squares. All together, this gives us a weight of

$$\begin{bmatrix} n \\ p \end{bmatrix} \begin{bmatrix} n-p \\ m-p \end{bmatrix}$$

as desired.

4.2 The q-Multinomial Coefficient

In general, we would like to have a way of handling tiles with an arbitrarily large color set. In the proof of Equation (4.2), we saw that the sum of the tilings of length n with p super-green squares, $m-p$ green squares, and $n-m$ red squares was

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ p \end{bmatrix} = \frac{[n]_q!}{[m]_q![n-m]_q!} \frac{[m]_q!}{[m-p]_q![p]_q!} = \frac{[n]_q!}{[n-m]_q![m-p]_q![p]_q!}$$

which looks similar to the multinomial coefficient $\binom{n}{n-m,m-p,q}$.

More generally, say we have c colors ranked from color 1 which is the "reddest" to color c which is the "greenest". We define our weighting scheme by saying that a tile counts all the tiles to its left of lower rank (i.e. all tiles to its left which are "redder" than it is.)

Suppose we want to create a tiling of length n using a_i tiles of color i where $\sum_{i=1}^{c} a_i = n$. We can start by placing the greenest tiles and working our way downward to the reddest tiles. The weights of the a_c greenest tiles will be $\begin{bmatrix} n \\ a_c \end{bmatrix}$. Once these are placed, we can ignore them and look at the next greenest tiles. The weights of the a_{c-1} second-greenest tiles placed on the remaining $n-a_c$ positions will be given by $\begin{bmatrix} n-a_c \\ a_{c-1} \end{bmatrix}$. Continuing in this fashion, we find that the total sum of the weights of the tilings created is

$$\begin{bmatrix} n \\ a_c \end{bmatrix} \begin{bmatrix} n-a_c \\ a_{c-1} \end{bmatrix} \cdots \begin{bmatrix} a_2+a_1 \\ a_2 \end{bmatrix} =$$

$$\frac{[n]_q!}{[a_c]_q![n-a_c]_q!} \frac{[n-a_c]_q!}{[a_{c-1}]_q![n-a_c-a_{c-1}]_q!} \cdots \frac{[a_2+a_1]_q!}{[a_2]_q![a_1]_q!} =$$

$$\frac{[n]_q!}{[a_c]_q![a_{c-1}]_q!\ldots[a_1]_q!}$$

In fact, this final formula is the algebraic definition of the q-multinomial coefficient $\begin{bmatrix} n \\ a_1, a_2, \ldots a_c \end{bmatrix}_q$. In the future, we will leave off the q subscript unless we have reason to include it.

Our combinatorial interpretation of the q-multinomial coefficient is that it gives the sum of the weights of the tilings of length n using c different colors ranked from lowest (reddest) to highest (greenest) with a_i tiles of color i.

4.3 A q-Analogue to Fermat's Little Theorem

Benjamin and Quinn [3] present a simple tiling proof to Fermat's Little Theorem, which states that for a prime p and any natural number a,

$$a^p \equiv a (\mathrm{mod}\ p) \tag{4.3}$$

Proof:
Question: How many ways are there (mod p) to create a tiling of length p using squares which come in a different colors?

Answer 1: For each of the p positions in the tiling, we have a choices for color, so our answer is a^p.

Answer 2: For each tiling which is not monochromatic, we place the tiling in the set of p tilings created by repeatedly removing the front tile and moving it to the back. Since p is prime, this procedure is guaranteed to give us p distinct tilings.

Thus, we can group all polychromatic tilings into groups of size p and all that remains (modulo p) are the a monochromatic tilings.

Thus,

$$a^p \equiv a (\mathrm{mod}\ p),$$

as desired.

The q-analogue proceeds along the same lines, but uses a different weighting scheme than we might expect. The q-identity we will prove is that for a prime p and a natural number a,

$$\sum_{k=1}^{p} \begin{bmatrix} p \\ k \end{bmatrix} \sum_{j=1}^{a} (a-j)^{p-k} \equiv a (\mathrm{mod}\ [p]_q) \tag{4.4}$$

Proof:
Question: What is the sum of the weights (mod $[p]_q$) of the tilings of length p using squares which come in a different colors?

Note that for this question we must define a different weighting scheme which can deal with the presence of a different colors. (The weighting scheme defined by the multinomial coefficient turns out to be insufficient for this theorem.) To do so, we label the a colors so that color 1 is the "reddest" and color a is the "greenest". To determine the weight of a tiling, we first look for the tile or tiles whose color is lowest ranked (i.e. the "reddest" tiles.) We then treat all tiles of this color as red and all other tiles as green and calculate the weight of the tiling using the usual method.

Answer 1: Let k count the number of tiles of lowest rank which appear in the tiling. For a fixed k, the sum of the weights of these tilings is $\begin{bmatrix} p \\ k \end{bmatrix}$ multiplied by the number of ways to distribute colors so that k tiles are of the lowest rank and $p-k$ tiles are of higher rank. If we fix the lowest rank at j, this means we have $a-j$ choices for each of the $p-k$ tiles which aren't of lowest rank. Thus, letting j and k vary, we find that the total sum of the weights of the tilings is

$$\sum_{k=1}^{p} \begin{bmatrix} p \\ k \end{bmatrix} \sum_{j=1}^{a} (a-j)^{p-k}.$$

Answer 2: Recall from the proof of Lemma 3.1 that if we take a polychromatic tiling of length p and create p distinct tilings by successively removing the first tile and placing it at the back, then the sum of the weights of these tilings will be a multiple of $[p]_q$. Despite the fact that our tiles now come in a colors rather than 2, the same lemma still holds since our weighting algorithm only considers a difference between the reddest tiles and all other tiles. Hence, we can group all polychromatic tilings into groups of size p where the sum of the weights of the tilings in each group is a multiple of $[p]_q$.

The only remaining tilings are the a monochromatic tilings, all of which have weight 1. Hence, our answer is simply a.

Thus,

$$\sum_{k=1}^{p} \begin{bmatrix} p \\ k \end{bmatrix} \sum_{j=1}^{a} (a-j)^{p-k} \equiv a (\mathrm{mod}\ [p]_q),$$

as desired.

Chapter 5

Alternating Sum Identities

In this chapter we will combine techniques from the previous chapters to examine several identities involving alternating sums.

5.1 A Tricolor Alternating Sum

This identity and its combinatorial proof were taken from Benjamin and Quinn's *Proofs that Really Count* [3].

$$\sum_k \binom{n}{k}\binom{k}{m}(-1)^k = (-1)^n \delta_{n,m} \tag{5.1}$$

Proof sketch:

Question: From the set of numbers 1 through n, give the alternating sum for the number of ways to select a subset of size k with a subsubset of size m. (Keep n and m fixed and alternate on the parity of k.)

Answer 1: We can do this directly by first selecting the subset of size k and then the subsubset of size m, giving us the alternating sum

$$\sum_k \binom{n}{k}\binom{k}{m}(-1)^k$$

Answer 2: If $n = m$, the only way we can select a subset of size $k = n$ with a subsubset of size m is by letting both the subset and the subsubset be the entire set of numbers 1 through n. This selection gets sign $(-1)^n$.

If $n \neq m$, we can associate each subset/subsubset selection with another selection where k has opposite parity. To do this, we simply consider the largest number not selected in the subsubset. (Since $n \neq m$, we are guaranteed to have at least one such number.) Toggling this number into and out of the subset of size k gives the desired bijection.

Thus, our answer is $(-1)^n$ if $n = m$ and 0 otherwise. That is, $(-1)^n \delta_{n,m}$.

The equivalent q-identity is

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^k = \tag{5.2}$$

$$\begin{cases} 0 & \text{if n-m is odd} \\ (-1)^n \begin{bmatrix} n \\ m \end{bmatrix} (1-q)(1-q^3)(1-q^5)\cdots(1-q^{n-m-1}) & \text{if n-m is even} \end{cases}$$

Proof:

Question: Give the alternating sum for the weights of the tilings of length n with $n-k$ red squares, $k-m$ green squares, and m super-green squares.

Answer 1: As in the standard binomial identity, we can calculate this as

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^k.$$

Answer 2: Note that, as we saw in Equation (4.2), instead of first choosing which k tiles aren't red we can choose which m tiles are super-green. These selections contribute weight $\begin{bmatrix} n \\ m \end{bmatrix}$. Having selected the super-green tiles, the problem is reduced to giving the alternating sum of ways to select $k-m$ green squares and $n-k$ red squares from the remaining $n-m$ positions. However, we have already solved this problem in the form of the Gaussian formula (3.4) and found an answer of

$$\begin{cases} 0 & \text{if n-m is odd} \\ (1-q)(1-q^3)(1-q^5)\cdots(1-q^{n-m-1}) & \text{if n-m is even} \end{cases}$$

We get an additional factor of $(-1)^n$ to account for whether the term in our sum with the largest value of k receives positive or negative weight. Putting these elements together, our final answer is

$$\begin{cases} 0 & \text{if n-m is odd} \\ (-1)^n \begin{bmatrix} n \\ m \end{bmatrix} (1-q)(1-q^3)(1-q^5)\cdots(1-q^{n-m-1}) & \text{if n-m is even} \end{cases}$$

as desired.

5.2 What if $q = -1$?

We already saw that if we let $q = 1$, we get $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$.

However, we may additionally want to consider the case where $q = -1$. I claim in this case that

$$\begin{bmatrix} n \\ k \end{bmatrix}_{-1} = \begin{cases} 0 & \text{if n is even, k is odd} \\ \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor} & \text{otherwise} \end{cases} \tag{5.3}$$

Proof:

Question: Of the tilings of length n using k green squares and $n-k$ red squares, what is the number whose weights have an even exponent minus the number whose weights have an odd exponent?

Answer 1: To count all the tilings positively, we would simply let $q = 1$ and take $\begin{bmatrix} n \\ k \end{bmatrix}_1$. However, if we instead let $q = -1$, we get all the tilings whose weights have an even exponent counted positively and all tilings whose weights have an odd exponent counted negatively. Thus, one answer is

$$\begin{bmatrix} n \\ k \end{bmatrix}_{-1}.$$

Answer 2: We'd like an involution which takes us from the set of tilings whose weights have even exponent to the set of tilings whose weights have odd exponent. Once we have an involution that works for almost all tilings, we'll count the remaining tilings for which it doesn't.

Divide the board into pairs of tiles. That is, look at the first two tiles as a unit, the third and fourth tiles as a unit, and continue in this fashion until the end of the board. (If n is odd, there will be one unpaired tile left at the end of the board.) Now find the first pair of tiles which is either rg or gr. Our involution will be to toggle this pair of tiles between rg and gr.

Note that this toggling doesn't change the interactions between the tiles in the pair and the remainder of the tiling. The r in the pair will still be counted by the same number of greens external to the pair and the the g will similarly count the same number of reds external to the pair. Thus, if this pair is rg and we change it to gr, the net effect on the tiling will be to decrease the weight by a factor of q since we are no longer counting the internal rg interaction. Similarly, toggling from gr to rg increases the weight by a factor of q. Thus, our involution always toggles the exponent on the weight between an even number and an odd number, as desired.

Now we need only consider the tilings which consist entirely of rr and gg pairs (with possibly one extra tile at the end). Note that the weight of any such tiling must have an even exponent. This is the case because each *pair* of red tiles contributes a weight of q^2 to each green square and a final unpaired red square contributes no weight, so we have no possibility for an odd exponent.

Thus it only remains to count the number of tilings with no rg or gr pairs. We will split our problem into cases depending on the parity of n and k.

- Case 1: n even, k odd
  Since n is even, we have no leftover tile at the end, so every tile is part of a pair. Furthermore, since k is odd, we can't possibly put every green tile in a gg pair. Thus, in this case, the number of tilings missed by the involution is 0.

- Case 2: n even, k even
  With both n and k even, we can create a tiling of entirely rr and gg pairs. Since we have $n/2$ pairs and $k/2$ of them are gg, the number of ways to do this is
  $$\binom{\frac{n}{2}}{\frac{k}{2}}$$

- Case 3: n odd, k odd
  With an odd n, we will have one unpaired tile at the end of our tiling. If the unpaired tile is red, the rest of the tiling is reduced to a situation of n even, k odd as in Case 1. Thus, we only have exceptions when the final tile is green. If this is the case, we are reduced to a Case 2 situation and therefore have
  $$\binom{\frac{n-1}{2}}{\frac{k-1}{2}}$$
  exceptions.

- Case 4: n odd, k even
  Again we will have an unpaired tile at the end. Since k is now even, if the unpaired tile is green we will be reduced to Case 1. If the unpaired

tile is red, we will be reduced to case 2 and thus have

$$\binom{\frac{n-1}{2}}{\frac{k}{2}}$$

exceptions.

Thus, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{-1} = \begin{cases} 0 & \text{if n is even, k is odd} \\ \left(\binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{k}{2} \rfloor}\right) & \text{otherwise} \end{cases}$$

as desired.

5.3 An Alternating-Sum Multinomial Identity

The following alternating sum identity, taken from Benjamin and Quinn's *Proofs that Really Count* [3], generalizes in a particularly interesting way. In particular, it involves the introduction of a multinomial coefficient even though the standard binomial identity uses only a binomial coefficient.

We present first a proof of the standard identity and then the q-generalization.

Theorem:

$$\sum_k \binom{n}{k} (n-k)^n (-1)^k = n! \tag{5.4}$$

Proof:

Question: Give the alternating sum for the number of ways to create a tiling of length n with k dark squares and $n-k$ light squares where every square in the tiling must point to a light square. (The sum should alternate on the parity of k.)

Note: I have intentionally chosen to use "light" and "dark" squares instead of red and green because the upcoming q-analogue will not assign weight based on the lightness or darkness of each tile but on a color which will be assigned later.

Answer 1: For a fixed k, first choose the locations of the k dark squares. (There are $\binom{n}{k}$ ways to do this.) Once the locations of the dark squares have been selected, there are $n-k$ light squares which may be pointed to, so we have an additional $n-k$ choices for each of the n squares. Thus, our alternating sum is $\sum\limits_k \binom{n}{k}(n-k)^n(-1)^k$.

Answer 2: We can create an involution to cancel out most of the terms in the alternating sum. Our involution will be to find the first tile which is not pointed to and toggle it between light and dark. The only exceptions which remain after this involution are the tilings in which every tile is pointed to. Since every tile is pointed to, these tilings must consist of all light squares ($k = 0$), so all of the exceptions will be counted positively in the sum. Furthermore, the number of ways to create a tiling in which every tile is pointed to by another tile is the same as the number of ways to order the numbers 1 through n. That is, our answer is simply $n!$.

Thus,

$$\sum_k \binom{n}{k} (n-k)^n (-1)^k = n!,$$

as desired.

I will now prove the following q-generalization: **Theorem:**

$$\sum_k \binom{n}{k} \sum_{a_i} \begin{bmatrix} n \\ a_1, a_2, \ldots a_{n-k} \end{bmatrix} (-1)^k = [n]_q! \tag{5.5}$$

Proof:

Question: Give the alternating sum for the weights of the tilings of length n with k dark squares and $n-k$ light squares where every square in the tiling must point to a light square. (The sum should alternate on the parity of k.)

The weighting scheme we use here is somewhat unusual. We allow our tiles to be any of n colors ranked from 1 (reddest) to n (greenest) in addition to being dark or light. The color assigned to a tile is given by the position of the tile it points to. For example, a tile which points to the tile on position 3 receives color #3. The weight of the tiling as a whole is given by having each tile count all tiles to its left of lower rank than itself. The darkness or lightness of a tile doesn't affect its weight in any way.

Answer 1: For a fixed k, first choose the locations of the k dark squares. (There are $\binom{n}{k}$ ways to do this.) Once the locations of the dark squares have been selected, we now have $n-k$ color choices for each of the n tiles. The sum of the weights of the tilings of length n using $n-k$ ranked colors is

$$\sum_{a_i} \begin{bmatrix} n \\ a_1, a_2, \ldots a_{n-k} \end{bmatrix},$$

so our alternating sum is

$$\sum_k \binom{n}{k} \sum_{a_i} \begin{bmatrix} n \\ a_1, a_2, \ldots a_{n-k} \end{bmatrix} (-1)^k.$$

Answer 2: As before, we can look for the first tile which is not pointed to and toggle it between light and dark. Note that this toggling has no effect on the weight of the tiling, so this works as a weight-preserving involution. The only exceptions are the tilings in which every tile is pointed to (i.e. the tilings which contain one tile of each color.) Our goal now is to determine the sum of the weights of the tilings of length n which use exactly one square of each color 1 through n.

To create such a tiling, we can choose the location of one square at a time, starting from the greenest square. If the greenest square is placed in the first position, it will get weight 1. In the second position, it will always count the one tile to its left and therefore get weight q. In general, in position i, the greenest square will get weight q^{i-1}, so the sum of the possible weights for the greenest square is $1 + q + \cdots + q^{n-1} = [n]_q$.

With the greenest tile placed, now choose where to place the second-greenest tile from the remaining $n-1$ positions. In the first of the available positions, it will get weight 1. In the second it will get weight q, and so forth. Altogether, the sum of the possible weights for the second-greenest tile is $1 + q + \cdots + q^{n-2} = [n-1]_q$.

Continuing in this fashion, we get that the sum of the possible weights for the $i+1$st greenest tile is $[n-i]_q$ so the sum of all the possible weights for the tiling is

$$[n]_q[n-1]_q \ldots [1]_q = [n]_q!$$

as desired.

Chapter 6

The Lattice Path Interpretation

As discussed in chapter 1, combinatorial proofs of q-identities usually make use of the lattice path interpretation for the q-binomial coefficients. It is natural to ask whether the tiling interpretation or the lattice path interpretation provides a clearer explanation of these identities. In that spirit, this chapter presents the lattice path proofs of several identities from previous chapters which work particularly well under the tiling interpretation.

6.1 Basic Identities

The identities presented in Section 2.2 translate fairly directly between the lattice path and tiling interpretations. I present them here with their proof sketches under the lattice path interpretation:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} \tag{6.1}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q^{n-k} \tag{6.2}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \tag{6.3}$$

$$\begin{bmatrix} 2n \\ n \end{bmatrix} = \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}^2 \tag{6.4}$$

Proof sketch for Equation 6.1: Transform a lattice path from $(0,0)$ to $(n-k,k)$ to a lattice path from $(0,0)$ to $(k,n-k)$ by flipping it along a line

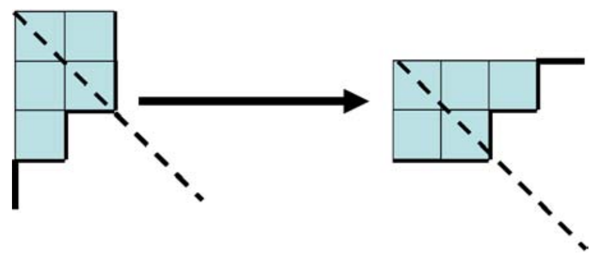


Figure 6.1: By reflecting this lattice path across the dotted line, we change its destination from $(2,4)$ to $(4,2)$ while preserving its weight.

through the upper left corner, as shown in Figure 6.1. (This is equivalent to the tiling interpretation in which we toggled the color of each square and reversed the order.)

Proof sketch for Equation 6.2:

Question: What is the sum of the weights of the lattice paths from $(0,0)$ to $(n-k,k)$?

Answer 1: By definition, $\begin{bmatrix} n \\ k \end{bmatrix}$.

Answer 2: Consider the direction of the last step in the path.

If the last step is right, then it contributes nothing to the weight of the lattice path, so the total weights of the paths of this form is just $\begin{bmatrix} n-1 \\ k \end{bmatrix}$.

On the other hand, if the last step is up, it must contribute a weight of q^{n-k} to the path and the total weight of the paths that can precede the final step is $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.

Thus, the total weight of the paths from $(0,0)$ to $(n-k,k)$ is

$$\begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

as desired.

Proof sketch for Equation 6.3:

Proceed in the same manner as the previous proof, but instead of looking at the last step in the path, consider the direction of the first step in the path.

Proof sketch for Equation 6.4:

Question: What is the sum of the weights of the lattice path from $(0,0)$ to (n,n)?
Answer 1: By definition,

$$\begin{bmatrix} 2n \\ n \end{bmatrix}$$

Answer 2: Consider the number of steps to the right out of the first n steps. Say j of the first n steps are to the right. Then the weight of the lattice path determined by just the first n steps is $\begin{bmatrix} n \\ n-j \end{bmatrix}$. If we temporarily ignore the first n steps of the path, the sum of the weights of the last n steps of the lattice paths is $\begin{bmatrix} n \\ j \end{bmatrix}$. However, there is also an interaction between the two portions of the path. This interaction is a square of side length j which lies above and to the left of the lattice path, as shown in Figure 6.2.

Hence, for a fixed j, the total weight of the lattice paths is

$$q^{j^2} \begin{bmatrix} n \\ n-j \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix}.$$

Finally, the weight of all desired paths is the sum over j. That is, our answer is

$$\sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ n-j \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix}.$$

By applying Equation (2.1), we get the desired identity:

$$\begin{bmatrix} 2n \\ n \end{bmatrix} = \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}^2$$

6.2 The q-Lucas' Theorem Revisited

In section 3.8, we saw a tiling-based proof of the theorem

$$\begin{bmatrix} pn+a \\ pk+b \end{bmatrix} \equiv \binom{n}{k} \begin{bmatrix} a \\ b \end{bmatrix} \pmod{[p]_q}$$

where p is prime and a and b are non-negative integers with $0 \leq a, b < p$.

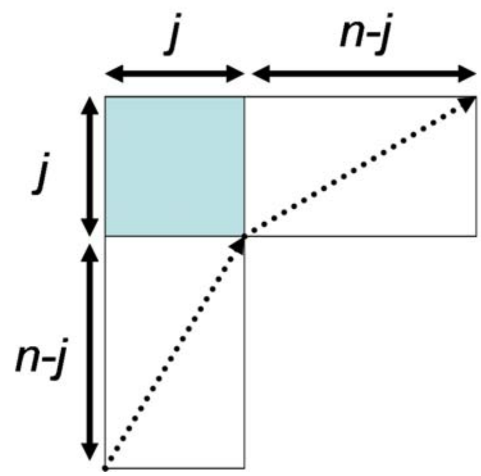


Figure 6.2: A lattice path from $(0, 0)$ to (n, n) with j right steps out of the first n. The first n steps of the path lie in the lower left rectangle and the final n steps lie in the upper right rectangle. Regardless of how we choose these paths, there is an additional area of j^2 above and to the right of the paths.

To translate this proof into lattice path language, we start with a new version of Lemma 3.1.

Lemma: Suppose we have a lattice path from $(0, 0)$ to (x, y) where x and y are both positive and $x + y = p$ for a prime p. If we create a set of p distinct lattice paths by successively removing the first step from the path and placing it at the end (and translating the resulting path either left or down so it still goes from $(0, 0)$ to (x, y)), then the sum of the weights of those paths will be divisible by $[p]_q$.

Proof: Recall that we are considering everything modulo $[p]_q$ and that $q^p \equiv 1$ modulo $[p]_q$.

Note that it is equivalent to show that the exponent on the weight of each of these p paths is distinct mod p. (Why? If each path has a different exponent (mod p), then their sum will be equivalent to $1 + q + q^2 + \cdots + q^{p-1}$ (mod $[p]_q$). Since $1 + q + q^2 + \cdots + q^{p-1} = [p]_q$, this shows that the sum of the weights is congruent to $[p]_q$, as desired.)

Now let's look at the effect on a path of moving the first step to the back. If the first step is up, moving it to the end will change the weight of the path by a factor of q^x since the up move contributed no weight at the start of the path, but at the end contributes weight equal to its x-coordinate. If, on the other hand, the first step is right, moving it to the back will change the weight of the tiling by a factor of q^{-y} since the right was previously increasing the x-coordinate of each up move by 1, and does not do so at the end of the path. However, since $q^p = q^{x+y} \equiv 1$, we also have $q^x \equiv q^{-y}$, so these actions have equivalent effects on the weight of the path (mod $[p]_q$).

Hence, if the weight of the first path is q^s, the weight of the second path is congruent to q^{s+x}. The weight of the third is congruent to q^{s+2x}, and so forth until the last tiling which has weight $q^{s+(p-1)x}$. However, since p is prime and x is strictly between 0 and p, these exponents are all distinct modulo p, as desired.

With this lemma in place, we can proceed to the proof of the q-Lucas' theorem.

Proof:

Question: What is the sum of the weights of the lattice paths (mod $[p]_q$) from $(0,0)$ to $(p(n-k)+(a-b), pk+b)$?

Answer 1: By definition,

$$\begin{bmatrix} pn+a \\ pk+b \end{bmatrix}.$$

Answer 2: Split the path up into regions. Consider the first p steps in the path as the first region, the next p steps as the second region, and so forth until the nth group of steps, which forms the nth region. The remaining a steps form the last region.

Look for the first region among the first n regions whose steps aren't all the same direction and perform the shifting procedure outlined in the lemma to create p different paths. (Leave the rest of the path the same, just permute the order of these p steps.) Our lemma showed that the sum of the weights of these paths will be a multiple of $[p]_q$. Furthermore, since permuting the steps within a region doesn't change the interactions between that region and the rest of the path, the sum of the weights of these p versions of the full lattice path is also a multiple of $[p]_q$. Hence, since we are working mod $[p]_q$, we need only consider the paths for which this shifting procedure is inapplicable, i.e. the paths in which each of the first n regions is comprised entirely of moves in the same direction.

To calculate the sum of the weights of these tilings, we will split them into two parts. First we consider the weight of the first n regions, then the weight of the final region, and finally the interactions between them.

We know the first n regions must have k regions of entirely up moves and $n-k$ regions of right moves. Since each region has p moves, we see that every region of right moves contributes weight q^{p^2} for each region of up moves after it. However, since we are working modulo $[p]_q$, we know $q^p \equiv 1$ and therefore $q^{p^2} \equiv 1$ as well. Hence, the weight of the first n regions will always be 1, so their contribution is simply the number of arrangements of the k regions of up moves and $n-k$ regions of right moves, i.e. $\binom{n}{k}$.

The sum of the weights of the final is easy to obtain. We have a path of length a with b up moves, so the sum of the weights is $\begin{bmatrix} a \\ b \end{bmatrix}$.

Finally, we must consider the interaction between the strip and the board. Each of the b up moves in the final region counts each of the $p(n-k)$ right moves in the first n regions, so we must multiply our answer of $\begin{pmatrix} n \\ k \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$ by an additional factor of $q^b p(n-k)$. However, modulo $[p]_q$, we have $q^{bp(n-k)} = (q^p)^{b(n-k)} = 1$. Thus, our answer is simply $\begin{pmatrix} n \\ k \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, as desired.

6.3 A Three-Color Identity Revisited

Recall that in the tiling interpretation, adding extra colors has very little effect on the complexity of the problem. That is, if we use red, green, and super-green squares instead of just red and green, it doesn't make the counting significantly more difficult. However, under the lattice path interpretation, this is not the case. Every additional color in the tiling interpretation corresponds to an extra dimension in the lattice-path interpretation. For example, recall Equation (4.2):

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ p \end{bmatrix} = \begin{bmatrix} n \\ p \end{bmatrix} \begin{bmatrix} n-p \\ m-p \end{bmatrix}$$

which we solved in Section 4.1 by considering a tiling with three colors.

To provide a combinatorial proof via lattice paths, we must extend our definition of the weight of a lattice path to the three-dimensional case. We do so by considering projections of the lattice path onto two-dimensional planes. For simplicity, we name the directions cardinally, so that the positive x direction is "east", the positive y direction is "north", and positive z direction is "up." Given a lattice path with unit steps in the east, north, and up directions, we calculate its weight in the following way:

- First, project the path onto the xy plane. (That is, look at our three-dimensional path from above.) This gives a two-dimensional lattice path. Calculate its weight in the standard way.
- Second, project the path onto the xz plane. (Look at the path from the south.) This gives another two-dimensional lattice path. Again, calculate the weight of this path in the standard way.
- Finally, project the path onto the yz plane. (Look at it from the east.) Calculate the weight of this two-dimensional projection.

The total weight of the lattice path is the product of the weights of these three projections.

With this weighting scheme in place, we are ready to provide a lattice-path combinatorial interpretation of our three-color tiling identity.

Lattice Path Proof of Equation 4.2:

Question: What is the sum of the weights of the lattice paths from $(0,0,0)$ to $(n-m, m-p, p)$ comprised of unit steps east, north, and up?

Answer 1: First choose the locations of the m north and up moves. Each step to the right contributes weight q to each north move in the xy-plane projection as well as contributing weight q to each up move in the xz-plane projection. Thus, we can select the locations of the m north and up moves at the same time. The total combined contribution of the two is $\begin{bmatrix} n \\ m \end{bmatrix}$. In fact, the locations of the north and up moves entirely determines the combined weights of the xy- and xz-plane projections. All that is left to determine is the interaction between the north and up moves to give us the weight of the yz-plane projection. To do so, choose which p of these m steps in this projection will be up steps. By definition, the number of ways to do this is $\begin{bmatrix} m \\ p \end{bmatrix}$. Thus, we have accounted for the weights of all three projections and our final answer is

$$\begin{bmatrix} n \\ m \end{bmatrix} \begin{bmatrix} m \\ p \end{bmatrix}.$$

Answer 2: First choose the location of the p up moves. If we look at the combined interactions of up moves with all other moves in the xz- and yz-plane projections, we see that the up moves get weight q for each non-up move preceding them. Hence, selecting the locations of the p up moves from the total n moves gives us the combined weights of the xz- and yz-plane projections as $\begin{bmatrix} n \\ p \end{bmatrix}$. Next, we must consider the weight of the xy-

plane projection. This is just a lattice path with $m - p$ north moves and $n - m$ east moves. This gives us a total weight of $\begin{bmatrix} n-p \\ m-p \end{bmatrix}$ for the xy-plane projection. Considering all three projections together, we obtain a weight of

$$\begin{bmatrix} n \\ p \end{bmatrix} \begin{bmatrix} n-p \\ m-p \end{bmatrix}$$

for the three-dimensional lattice paths, as desired.

This proof is far less intuitive than the one using the tiling interpretation, and requires much more complicated visualization. Furthermore, if we wanted to extend this process to dimensions higher than three, the number of projections we consider would grow very quickly, not to mention the additional difficulty of visualizing nonspatial dimensions.

Chapter 7

Conclusion and Future Work

7.1 Future Work

In my research, I primarily searched for either known q-identities which lacked satisfactory combinatorial proofs or binomial identities with known combinatorial proofs which I might try to translate into q-language. Both these categories contain a vast number of identities, and although I accomplished much this semester, there are still many possible avenues for future research.

Several identities stand out as strong candidates for combinatorial proof under my tiling interpretation:

Jacobi's Triple Product Identity:

$$\sum_{n=-\infty}^{\infty} z^n q^{\binom{n+1}{2}} = \prod_{n=1}^{\infty} (1-q^n)(1+zq^n)(1+z^{-1}q^{n-1})$$

This well-known identity has a relatively simple algebraic proof given in Andrews and Eriksson [2], but seems likely to have a nice combinatorial interpretation as well.

***q*-Wilson's Theorem:** If $p > 3$ is a prime and $p \equiv 3 \pmod 4$, then

$$\prod_{j=1}^{p-1} [j]_{q^j} \equiv -1 \pmod{[p]_q}.$$

This identity is presented in a paper by Robin Chapman and Hao Pan [4] and proven using concepts from abstract algebra including field extensions and automorphisms. However, the simple form of the equation suggests that a combinatorial explanation might be possible.

Sums of q-quarts and quints: I presented q-analogues to the identities on the sums of consecutive integers, integer squares, and cubes. Michael Schlosser [11] has also provided q-generalizations for the sums of fourth and fifth powers. Their current forms are rather hard to interpret from a combinatorial standpoint, but it is likely that they could be converted to forms which more closely resemble the standard binomial identities with an additional error term of $(q-1) \cdot f(q)$ which would disappear when $q = 1$.

There are also several concepts which I presented in this paper but didn't have a chance to fully explore. For example, I gave formulas for q-multichoose and q-multinomial coefficients but only used them in one proof each. A good warm-up for a reader interested in conducting research into q-identities might be to seek out multichoose and multinomial identities and attempt to find their q-generalizations.

7.2 Strengths of the Tiling Interpretation

In the previous chapter, I briefly discussed the strengths of the tiling interpretation I introduce in this paper as compared to the canonical lattice-path interpretation. In particular, we saw how certain proofs can be completed with the addition of extra colors in the tiling interpretation but may require the addition of extra dimensions when working with lattice paths.

More generally, I have found that my tiling interpretation is easier to work with because it more formulaic than the lattice-path interpretation. The effect of, say, adding a green tile to the end of a tiling is immediately obvious because of the way in which we defined our weighting function. The corresponding addition of an up move to the end of a lattice path may require us to think a little more to determine how that changes the weight of the path. By thinking of everything in terms of tilings and considering the weight of each tile, we have distilled the essence of the weighting scheme. On the other hand, in doing so, we have taken away some of the versatility of the lattice-path interpretation.

In my opinion, the trade-off is a worthwhile one. By considering q-binomial coefficients as sums of weighted tilings, we are able to strike to the heart of their meaning with a simple and direct weighting algorithm as well as always limiting ourselves to an easy-to-visualize, one-dimensional case.

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