A Decimation-in-Frequency Fast-Fourier Transform for the Symmetric Group

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Abstract

A Discrete Fourier Transform (DFT) changes the basis of a group algebra from the standard basis to a Fourier basis. An efficient application of a DFT is called a Fast Fourier Transform (FFT). This research pertains to a particular type of FFT called Decimation in Frequency (DIF). An efficient DIF has been established for commutative algebra; however, a successful analogue for non-commutative algebra has not been derived. However, we currently have a promising DIF algorithm for $\mathbb{C}S_n$ called Orrison-DIF (ODIF). In this paper, I will formally introduce the ODIF and establish a bound on the operation count of the algorithm.

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Chapter 1

Introduction

1.1 Motivation and Outline of the Project

Let M be a 5-dimensional vector space over $\mathbb{C}$, and let A be a linear transformation acting on M. We would like to apply A to elements in M efficiently. Clearly, A can be written as a 5×5 matrix. However, if we choose a basis of M blindly, the matrix representation of A might be a full matrix. Therefore, in the worst case, it takes 5×5 multiplications and 4×5 additions to apply A to an element in M. Let S be such a blindly chosen basis, and denote the matrix representation of A in the basis S by $[A]_S$. Then, for $v \in M$, $[A]_S[v]_S$ might look like the following:

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix} \begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix}$$

Figure 1.1: Worst case scenario: $[A]_S[v]_S$.

Suppose, however, that there exists a basis β of M that can be partitioned into subsets $\beta_1, \beta_2, \ldots \beta_\ell$ in such a way that the subspace spanned by each β_i is closed under (invariant under) the action of A. For now, let $\ell = 3$, $|\beta_1| = 1$, and $|\beta_2| = |\beta_3| = 2$. Therefore M decomposes into a direct sum of three invariant spaces:

$$M = span(\beta_1) \oplus span(\beta_2) \oplus span(\beta_3).$$

Then $[A]_\beta$ is a block diagonal matrix, with each block having dimension $|\beta_i| \times |\beta_i|$. Hence $[A]_\beta[v]_\beta$ would look like the following:

$$\begin{pmatrix} \bullet & & & & \\ & \bullet & \bullet & & \\ & \bullet & \bullet & & \\ & & & \bullet & \bullet \\ & & & \bullet & \bullet \end{pmatrix}\begin{pmatrix}\bullet\\ \bullet\\ \bullet\\ \bullet\\ \bullet\end{pmatrix} \quad \begin{matrix} \left.\vphantom{\bullet}\right\}\beta_1 \\ \left.\begin{matrix}\bullet\\ \bullet\end{matrix}\right\}\beta_2 \\ \left.\begin{matrix}\bullet\\ \bullet\end{matrix}\right\}\beta_3 \end{matrix}$$

Figure 1.2: $[A]_\beta[v]_\beta$.

Under β, one can thus apply A to any element of M with at most

$$\underbrace{1}_{\beta_1} + \underbrace{2 \times 2}_{\beta_2} + \underbrace{2 \times 2}_{\beta_3} = 9 \qquad \text{multiplications and}$$

$$\underbrace{0}_{\beta_1} + \underbrace{1 \times 2}_{\beta_2} + \underbrace{1 \times 2}_{\beta_3} = 4 \qquad \text{additions.}$$

Thus the existence of a basis like β is good news. Because β respects M's decomposition into spaces that are invariant under the action of A, one can analyze A's action locally. It should be clear that the smaller $|\beta_i|$ is for each i, the smaller the number of operations that an application of A to an arbitrary vector requires. Hence it is indeed best if each β_i spans the finest invariant subspace possible. We call an invariant space that does not contain any proper invariant subspace an **irreducible space**.

If A is diagonalizable under some other basis β', then each vector of β' spans a 1-dimensional vector space invariant under A (also known as eigen-space). A 1-dimensional invariant subspace is indeed irreducible, so β' is a basis that respects M's decomposition into irreducible subspaces.

Suppose, however, that we want to analyze the actions of two linear transformations A and B simultaneously. Then we would like to use a basis that respects a decomposition of M into subspaces that are irreducible under the actions of both A and B. In our research, we consider the action of not just one or two linear transformations acting on a vector space, but many more. In fact, we will be considering the action of all of the elements in a group algebra $\mathbb{C}G = \{\sum_{g \in G} c_g g; c_g \in \mathbb{C}, g \in G\}$.

Let us define a $\mathbb{C}G$-module. This definition can be found in Dummit and Foote (1991). A $\mathbb{C}G$-module is a set M together with

1. a binary operation $'+'$on M under which M is an abelian group,
2. an action $'\bullet'$ of $\mathbb{C}G$ on M that satisfies
   (a) for all $a \in \mathbb{C}G$ and $m, n \in M$, $a \bullet (m+n) = a \bullet m + a \bullet n$
   (b) for all $a, b \in \mathbb{C}G$ and $m \in M$, $(a+b) \bullet m = a \bullet m + b \bullet m$ and $a \bullet (b \bullet m) = ab \bullet m$.
   (c) for all $c \in \mathbb{C}$, $c \bullet m = cm$.

Note that M is a vector space over $\mathbb{C}$. Also, notice that if $g \in G$, a map $D_g : m \mapsto g \bullet m$ is a $\mathbb{C}$-linear transformation on M. Thus, under any basis γ of M , D_g can be expressed as a matrix $[D_g]_\gamma$ of dimension $dim(M) \times dim(M)$ over $\mathbb{C}$. The map $D_M : g \mapsto [D_g]_\gamma$ is called a **representation of** G. If N is another $\mathbb{C}G$-module that is isomorphic to M, then there exist bases in M and N such that $D_M = D_N$. If M is irreducible, we call D_M an **irreducible representation of** G. Because the representation of G completely determines the structure of the $\mathbb{C}G$-module, sometimes the module itself is called a representation of $\mathbb{C}G$. In order to avoid confusion, however, we do not use the term 'representation.' this way.

$\mathbb{C}G$ is a vector space with the standard basis that consists of the elements in G. Our research considers the case in which $G = S_n$, and $M = \mathbb{C}S_n$. In particular, suppose that we want to study the linear transformations induced by the actions of $\mathbb{C}S_n$ on the elements of $\mathbb{C}S_n$. Thus we want to use a basis that respects $\mathbb{C}S_n$'s decomposition into subspaces that are irreducible under the action of all elements in $\mathbb{C}S_n$. We call a $\mathbb{C}S_n$-invariant subspaces a $\mathbb{C}S_n$-submodule; we call it a left(right) $\mathbb{C}S_n$-module if the action is defined from left(right)[1] The basis that we described above is therefore **a basis that respects $\mathbb{C}S_n$'s decomposition into irreducible $\mathbb{C}S_n$-modules (also called $\mathbb{C}S_n$-irreducibles).** Such a basis is called a **Fourier basis**. Fortunately, Fourier bases exist for any group algebra $\mathbb{C}G$. **The Fourier transform** is a change of basis from the standard basis to a Fourier basis. When

[1]In most cases, facts that apply to left $\mathbb{C}S_n$-modules also apply to right $\mathbb{C}S_n$-modules. Therefore throughout this chapter, we will focus on left $\mathbb{C}S_n$-modules.

G is finite, we call the transform a **discrete Fourier transform (DFT)**, and call the change of basis matrix a DFT matrix.

However, in real world application of $\mathbb{C}S_n$, raw data is often in the standard basis (Hansen, 2007). Hence the existence of a Fourier basis is not so helpful if the change of basis from the standard basis to a Fourier basis requires so many operations that we are better off studying the action of $\mathbb{C}S_n$ in the standard basis. Therefore , we would like to implement the change of the basis to a Fourier basis as efficiently as possible. An efficient application of a DFT is called a **fast Fourier transform (FFT.)**

Fast Fourier Transform

One way to apply a DFT efficiently is to factor the DFT matrix into multiple sparse matrices. This amounts to changing the basis in steps; each factor in the factorization of the DFT matrix will correspond to an **intermediate basis** of the FFT.

The theory of FFTs has its origin in the field of signal processing, in which $G = C_n$. Let ω denote a primitive nth root of unity $e^{2\pi i/n}$. Suppose that $< x >= C_n$, and let the standard basis be $S = \{1, x, \dots x^{n-1}\}$. Then consider a vector $\sum_{t=1}^{n-1} v(t)x^t$ in $\mathbb{C}C_n$. We apply a DFT matrix of $\mathbb{C}C_n$ to $[\sum_{t=1}^{n-1} v(t)x^t]_S$ to obtain the vector form of $\sum_{t=1}^{n-1} v(t)x^t$ in Fourier basis.

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^k & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2k} & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^f & \omega^{2f} & \vdots & \omega^{kf} & \vdots & \omega^{(n-1)f} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \dots & \omega^{k(n-1)} & \dots & \omega^{(n-1)^2} \end{pmatrix} \begin{pmatrix} v(0) \\ v(1) \\ v(2) \\ \vdots \\ v(f) \\ \vdots \\ v(n-1) \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_f \\ \vdots \\ c_{n-1} \end{pmatrix}$$

Figure 1.3: $[DFT_{C_n}][\sum_{t=1}^{n-1} v(t)x^t]_S$.

We see that

$$c_f = \sum_{t=0}^{n-1} v(t)\omega^{ft}. \tag{1.1}$$

We can obtain the factorization of the DFT matrix by breaking down the computation of $c_1, \ldots c_{n-1}$. There are two ways to break down the computation. **Decimation in** f decimates f by writing f as $f_1 + f_2 m$ for some integer m that divides n. **Decimation in** t, on the other hand, decimates t by writing t as $t_1 + t_2 s$ for some integer s that divides n and directly breaking down the expression of (1.1). Rockmore (2004) and Gentleman and Sande (1966) discuss this more extensively. From the terminology of the Fourier transform in signal processing, the vector space $\mathbb{C}C_n$ under a Fourier basis is called the **frequency domain**, and the same space under the standard basis is called the **time domain**. So each f respresent a coordinate in the frequency domain, and each t represent a coordinate in the time domain. Therefore we call the first way of FFT as **decimation-in-frequency (DIF)** and the latter as **decimation-in-time (DIT)** .

The purpose of this research is to compute the runtime of a certain type of decimation in frequency FFT called the Orrison-DIF (ODIF) on $\mathbb{C}S_n$. The ODIF is a promising algorithm which was pioneered by Michael Orrison. However, the mechanism of the algorithm has never been described formally. In Chapter 2, I will provide a rigorous presentation of the ODIF. In Chapter 3, I will present all of the structural facts about $\mathbb{C}S_n$ that are required in computing the runtime of the ODIF. The last chapter uses the tools built in Chapter 3 to provide the formula for the operation count of the ODIF.

I will conclude this chapter with the tools required for understanding Chapter 2.

1.2 Key Tools in Analyzing FFTs

In this section, I will present Wedderburn's Theorem, which allows us to see the Fourier Transform as a ring homomorphism. Most of the key ideas presented here can be found in Clausen and Baum (1993). Let us begin with some theorems that are instructive in understanding the general strucuture of $\mathbb{C}G$. As mentioned in Section 1.1, a Fourier basis exists for every $\mathbb{C}G$. That is, there exists a basis β of $\mathbb{C}G$ that can be partitioned into subsets $\beta_1, \ldots \beta_\ell$ such that each β_i spans an irreducible $\mathbb{C}G$-module. Thus $\mathbb{C}G$ decomposes into a direct sum of irreducible $\mathbb{C}G$-modules. In fact, this applies to any $\mathbb{C}G$-module:

Theorem 1.1. *(Maschke) Every $\mathbb{C}G$-module is a direct sum of irreducible $\mathbb{C}G$-modules.*

Next, consider a representation D of G corresponding to M. If β is a basis that respects the decomposition of M into irreducible $\mathbb{C}G$-modules, $[D(g)]_\beta$ is clearly a block diagonal matrix, where each block in the diagonal is the representation of G in an irreducible $\mathbb{C}G$-module contained in M. Therefore, the above statement is equivalent to saying that **every representation of G is equivalent to a direct sum of irreducible representations of G.**

Some left $\mathbb{C}G$-modules might contain multiple irreducible left $\mathbb{C}G$-modules that are isomorphic to each other. We call the span of all left isomorphic irreducible submodules in a given left $\mathbb{C}G$-module a **left $\mathbb{C}G$-isotypic space**. The decomposition of any left $\mathbb{C}G$-module M into left $\mathbb{C}G$-isotypic space is unique (Clausen and Baum, 1993). In particular, the following applies to $\mathbb{C}G$.

Theorem 1.2.

- *$\mathbb{C}G$ is the direct sum of minimal two-sided ideals: $\mathbb{C}G = \oplus_j^h I_j$. Each I_j is a $\mathbb{C}G$-isotypic space both under the right and left actions of $\mathbb{C}G$.*
- *The decomposition of $\mathbb{C}G$ into minimal two-sided ideals is unique.*
- *If $1 = e_1 + \cdots + e_h$ with $e_j \in I_j$ then the e_js are pairwise orthogonal primitive idempotents in the center of $\mathbb{C}G$ (centrally primitive idempotents). Moreover, e_j is a unit element in the algebra I_j.*
- *Every minimal left ideal (left irreducible $\mathbb{C}G$-module) L is contained in exactly one I_j. If the dimension of L is d_i, I_i is a direct sum of d_i mutually isomorphic left ideals.*
- *Every isomorphism type of an irreducible $\mathbb{C}G$-module is present in $\mathbb{C}G$.*

Again, for details, consult Clausen and Baum (1993). The implication of the second and the third part of the theorem is significant. They indicate that the centrally primitive idempotent e_i acts as an identity in the irreducible left $\mathbb{C}G$-modules contained in I_i, and acts as zero in the left irreducible $\mathbb{C}G$-module contained in I_j if $i \neq j$. In particular, this means that e_i acts as an identity in any left irreducible module isomorphic to the left irreducible $\mathbb{C}G$-modules contained in I_i, and acts as zero in any left irreducible $\mathbb{C}G$-module of different isomorphism type. Thus, for any left $\mathbb{C}G$-module M, e_iM is a $\mathbb{C}G$-isotypic space in M containing left irreducible $\mathbb{C}G$-modules isomorphic to the irreducible $\mathbb{C}G$-modules in I_i. **The action of e_i from the left projects the elements in $\mathbb{C}G$ to the left $\mathbb{C}G$-isotypic space correponding to e_i.**

It should be noted that, although the decomposition of a $\mathbb{C}G$-modules into isotypic subspaces is unique, its decomposition into irreducible $\mathbb{C}G$-modules may not be unique. Consider, for example, the action of $\mathbb{C}\mathbb{Z}_2$ on a 3-dimentional vector space V over $\mathbb{C}$. Let x be the generator of $\mathbb{Z}_2$. Then $D : \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}^{3\times 3}$ defined by

$$D(id) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad D(x) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

is a representation of $\mathbb{C}\mathbb{Z}_2$. $\mathbb{C}^3$ then decomposes into two $\mathbb{C}\mathbb{Z}_2$-isotypic spaces: the 1-dimensional space N_1 that is spanned by $[1,0,0]^T$, and the 2-dimensional space N_2 that is spanned by $[0,1,0]^T$ and $[0,0,1]^T$. It is clear that this isotypic decomposition is unique. However, V can be decomposed into three 1-dimensional irreducible spaces in infinitely many ways. In particular, for any choices of two vectors v_1, v_2 spanning I_2, $span\{v_1\}$ and $span\{v_2\}$ are both 1-dimensional irreducible spaces. The decomposition into irreducible spaces can be unique if each isotypic space is an irreducible space itself.

1.2.1 Wedderburn's Theorem

For the theorems that appear in this subsection, consult Clausen and Baum (1993) for proofs.

The motivation behind a discrete Fourier transform was a Fourier basis—a basis under which the analysis of the left regular action of $\mathbb{C}G$ on $\mathbb{C}G$ is easy. The regular action is indeed a proper module action. Therefore, just like any other module action, the regular action is associated with a representation. This representation is called the *regular representation.* We denote the left regular representation by D_ρ. Let us look at what D_ρ looks like. We saw in the Theorem 1.2 that

$$\mathbb{C}G \cong \bigoplus d_i L_i$$

where each L_i is a left irreducible $\mathbb{C}G$-module contained in I_i, and d_i is the dimension of L_i. Suppose that D_i is a representation of G corresponding to L_i. We see that there is a Fourier basis β of $\mathbb{C}G$ so that

$$D_\rho = \bigoplus d_i D_i.$$

For example, consider $\mathbb{C}S_3$. $\mathbb{C}S_3 = L_1 \oplus L_2 \oplus L_2' \oplus L_3$, where L_1, L_3 are non-isomorphic 1-dimensional left irreducible $\mathbb{C}S_3$-modules, and L_2, L_2' are isomorphic 2-dimensional left irreducible $\mathbb{C}S_3$-modules. Thus, under a specific Fourier basis under which the representation of $\mathbb{C}S_3$ for L_2 and the representation of $\mathbb{C}S_3$ for L_2' are equal, D_ρ for $\mathbb{C}S_3$ is

$$D_1 \oplus D_2 \oplus D_2 \oplus D_3$$

where D_1 is a representation of S_3 for L_1, D_2 is a representation of S_3 for L_2 and L_2', and D_3 is a representation of S_3 for L_3. Let $v, a \in \mathbb{C}S_3$, and denote ijth entry of $[D_k(a)]_\beta$ by $D_k(a)_{ij}$. Then $[D_\rho(a)]_\beta[v]_\beta$ looks like the following:

$$\begin{pmatrix} D_1(a) & & & & & \\ & D_2(a)_{11} & D_2(a)_{12} & & & \\ & D_2(a)_{21} & D_2(a)_{22} & & & \\ & & & D_2(a)_{11} & D_2(a)_{12} & \\ & & & D_2(a)_{21} & D_2(a)_{22} & \\ & & & & & D_3(a) \end{pmatrix} \begin{pmatrix} \diamondsuit \\ \spadesuit \\ \spadesuit \\ \clubsuit \\ \clubsuit \\ \heartsuit \end{pmatrix} \begin{array}{l} \} L_1 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} L_2 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} L_2' \\ \} L_3 \end{array}$$

Figure 1.4: $[D_\rho(a)]_\beta[v]_\beta$.

Note that the above is equivalent to

$$\begin{pmatrix} D_1(a) & & & \\ & D_2(a)_{11} & D_2(a)_{12} & \\ & D_2(a)_{21} & D_2(a)_{22} & \\ & & & D_3(a) \end{pmatrix} \begin{pmatrix} \diamondsuit & & & \\ & \spadesuit & \clubsuit & \\ & \spadesuit & \clubsuit & \\ & & & \heartsuit \end{pmatrix}.$$

Thus, under a Fourier basis, we can realize the multiplication of two elements in $\mathbb{C}S_3$ as a multiplication of two matrices. This, in fact holds in general:

Theorem 1.3. *(Wedderburn) As rings,*

$$\mathbb{C}G \cong \bigoplus_{i=1}^{h} C^{d_i \times d_i} \tag{1.2}$$

where h is the number of two-sided ideals of $\mathbb{C}G$. Moreover, each $\mathbb{C}^{d_i \times d_i}$ is isomorphic to I_i in Theorem 1.2, and each column/row of $\mathbb{C}^{d_i \times d_i}$ is isomorphic to a left/right irreducible $\mathbb{C}G$-module.

Thus a Fourier transform is not only a change of basis, but a ring isomorphism. Following the convention of the signal processing, we say we view the elements of $\mathbb{C}G$ in the **time domain** when we view them in the standard basis of $\mathbb{C}G$. Also, we say we view the elements of $\mathbb{C}G$ in the **frequency domain** when we view them in the matrix basis of the right-hand-side of (1.2). For any isomorphism ϕ between $\mathbb{C}G$ and $\bigoplus_{i=1}^{h} C^{d_i \times d_i}$, the preimage of the matrix basis under ϕ is a Fourier basis, because each column/row of $\mathbb{C}^{d_i \times d_i}$ is isomorphic to a left/right irreducible $\mathbb{C}G$-module. Thus the isormorphism map ϕ determines a Fourier basis. In this research, we will be looking at a Fourier basis with a specific property:

Definition : *Suppose T is a chain of subgroups $G_0 \leq G_1 \leq \cdots \leq G_n = G$. We say that a basis β of $\mathbb{C}G$ is right (left)-adapted to a subgroup chain T, if for any $i = 1, \ldots n$, β can be partitioned into subsets such that each subset spans a distinct irreducible $\mathbb{C}G_i$-module. Also, given another chain of subgroups S, we say that a basis is (T, S)-doubly-adapted if the basis is both left-adapted to T and right-adapted to S.* [2] *Also, we say that a basis β is left weakly adapted to a chain T if β can be partitioned into subsets such that each subset spans a distinct left $\mathbb{C}G_k$ isotypic space for all G_k in the chain. If a basis is left weakly adapted to a chain T and right weakly adapted to chain S, we say that the basis is (T, S)-weakly adapted.*

Fortunately,

Theorem 1.4. *(The existence of adapted basis) For any $\mathbb{C}G$-module, a T-adapted basis exists for any subgroup chain T of G.*

Clausen and Baum (1993) provides a proof for this theorem. In ODIF, we will focus our attention on the specific Fourier basis that is doubly adapted to the chain of subgroups $S_1 \leq S_2 \leq \cdots \leq S_n$. We will construct this basis from a series of intermediate bases that are weakly adapted to shorter subgroup chains. We will discuss how we can achieve this in the next chapter.

[2] In fact, for any G there exists a basis that not only respects $\mathbb{C}G$'s decomposition into irreducible $\mathbb{C}G_i$-modules for each G_i in the given chain of subgroups, but also has a property that $\mathbb{C}G$'s representations corresponding to any isomorphic irreducible $\mathbb{C}G_i$-modules under that basis are equal. (Clausen and Baum, 1993) refers to a basis with this property as an *adapted basis.* However, in our research, we will not exploit this second property.

Chapter 2

Understanding the Orrison-DIF

Suppose $\{W_i\}$ is the complete set of the left $\mathbb{C}S_n$-isotypic spaces in $\mathbb{C}S_n$. If N is a minimal left irreducible $\mathbb{C}S_n$-module in W_i, denote $dim(N)$ by d_i. From Wedderburn's Theorem,

$$\mathbb{C}S_n \cong \bigoplus_i \mathbb{C}^{d_i \times d_i}.$$

Suppose ϕ is an isomorphism map for the above isomorphism. Then each $\mathbb{C}^{d_i \times d_i}$ is the image of W_i under ϕ. Each column of $\mathbb{C}^{d_i \times d_i}$ is isomorphic to N, and each row of $\mathbb{C}^{d_i \times d_i}$ is isomorphic to a right $\mathbb{C}S_n$ irreducible module of dimension d_i in W_i. The action of $\mathbb{C}S_n$ on $\phi(N)$ is defined by the representation $\phi_i : \mathbb{C}S_n \to End_{\mathbb{C}}(N)$, which is a map from $\mathbb{C}S_n$ to $\mathbb{C}^{d_i \times d_i}$.

The discrete Fourier transform is a change of basis in $\mathbb{C}S_n$ from the standard basis to a basis that can be partitioned into subsets such that each subset spans a left $\mathbb{C}S_n$-irreducible space. Because each W_i is a left $\mathbb{C}S_n$-isotypic space, such a basis can indeed be partioned into sets $\{B_i\}$, where each B_i spans W_i. Thus a DFT can also be envisioned as a projection onto the W_is, or left $\mathbb{C}S_n$-isotypic spaces. Projecting $\mathbb{C}S_n$ onto W_i is easy– one must simply multiply each element in $\mathbb{C}S_n$ on the left by the centrally primitive idempotent e_i corresponding to W_i. On the other hand, the projection into right-isotypic spaces can be achieved by multiplying the corresponding idempotents from the right.

However, the projection onto left-$\mathbb{C}S_n$-isotypic spaces is not enough complete the change of basis. In particular, a projection merely decimates the $n!$-dimensional space $\mathbb{C}S_n$ into d_i^2-dimensional spaces, and we have to decide the basis within each d_i^2-dimensional space so that it respects the $\mathbb{C}S_n$-isotypic spaces' decomposition into $\mathbb{C}S_n$-irreducible modules. Because there are aninfinite number of ways to decompose multi-dimensional $\mathbb{C}S_n$-isotypic spaces into $\mathbb{C}S_n$-irreducible modules, this can be a problem. In order to eliminate this arbitrariness, we choose a specific Fourier basis and project $\mathbb{C}S_n$ into 1-dimensional spaces spanned by each vector in the chosen basis.

The Fourier basis of $\mathbb{C}S_n$ that we will aim for is the basis that is doubly-adapted to the chain of subgroups

$$S_1 \leq S_2 \leq \cdots \leq S_n.$$

Recall that a basis that is doubly adapted to the chain above respects the decomposition of $\mathbb{C}S_n$ into left $\mathbb{C}S_k$-irreducibles for all k. Therefore, a ϕ associated with this basis has a following property:

The entries in

$$\phi(\mathbb{C}S_n) = \bigoplus_i \mathbb{C}^{d_i \times d_i}$$

can be partitioned not only into sets such that the preimage of each set spans a left $\mathbb{C}S_n$-irreducible, but also, for all k, into sets such that the preimage of each set spans a left $\mathbb{C}S_k$-irreducible of $\mathbb{C}S_n$.

Throughout this chapter, we will consider $\mathbb{C}S_n$ in the frequency domain under such a ϕ. We will achieve the doubly adapted basis by projecting $\mathbb{C}S_n$ into each entry in $\phi(\mathbb{C}S_n) = \bigoplus_i \mathbb{C}^{d_i \times d_i}$. We will take advantage of a specific property of $\mathbb{C}S_n$, and use series of projections into isotypic space(i.e. isotypic projections) to decimate $\mathbb{C}S_n$ into 1-dimensional spaces. Let us explain how we can do this.

2.1 Decimation in Frequency and the ODIF

For $k \leq n$, a left $\mathbb{C}S_n$-module is a left $\mathbb{C}S_k$-module; therefore a left $\mathbb{C}S_n$-irreducible module is decomposable into left $\mathbb{C}S_k$-irreducible modules. In the language of ϕ, this implies that the preimage of each column in $\mathbb{C}^{d_i \times d_i}$ is decomposable into left $\mathbb{C}S_k$-irreducible modules. Thus, for each i, we shall be able to partition the set of entries in each column of $\mathbb{C}^{d_i \times d_i}$ into subsets such that each subset spans ϕ's image of left $\mathbb{C}S_k$-irreducible modules. Suppose, for example, W_a is a 9-dimensional left $\mathbb{C}S_n$-isotypic space containing three left $\mathbb{C}S_n$-irreducible modules of dimension 3. Further suppose that a $\mathbb{C}S_n$-irreducible M in W_a decomposes into a 1-dimensional left

$\mathbb{C}S_k$-irreducible N_1 and a 2-dimensional left $\mathbb{C}S_k$-irreducible N_2. So we can write

$$M = N_1 \oplus N_2.$$

Then we shall be able to partition 3 entries in the column $\phi(M)$ into one entry corresponding to N_1, and two entries corresponding to N_2. (Fig 2.1). Under ϕ, this is equivalent to decimating the 3×3 matrix by rows (Fig 2.2).

$$\begin{pmatrix} \bullet \\ \bullet \\ \bullet \end{pmatrix} \begin{array}{l} \} \quad N_1 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad N_2 \end{array}$$

Figure 2.1: $\phi(M)$.

$$\begin{pmatrix} \circ & \circ & \circ \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \begin{array}{l} \} \quad R_1 : \mathbb{C}S_k\text{-isotypic in containing } N_1 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad R_2 : \mathbb{C}S_k\text{-isotypic containing } N_2 \end{array}$$

Figure 2.2: $\phi(W_a)$.

Denote the preimage of the top row and the bottom two rows under ϕ by R_1 and R_2, respectively. Each subcolumn of R_i in Figure 2.2 is a left $\mathbb{C}S_k$-irreducible module that is isomorphic to N_i. The representation $D_a : \mathbb{C}S_n \mapsto \mathbb{C}^{3\times 3}$ of the left regular action of $\mathbb{C}S_n$ in M decomposes into a 1-dimensional representation $E_1 : \mathbb{C}S_k \mapsto \mathbb{C}$ of the left regular action of $\mathbb{C}S_k$ on the left $\mathbb{C}S_k$-irreducibles of R_1, and a 2-dimensional representation $E_2 : \mathbb{C}S_k \mapsto \mathbb{C}^{2\times 2}$ of the left regular action of $\mathbb{C}S_k$ on the left $\mathbb{C}S_k$-irreducibles of R_2. The restriction $D_a \downarrow \mathbb{C}S_k$ looks like the following:

$$\begin{pmatrix} \bullet & & \\ & \bullet & \bullet \\ & \bullet & \bullet \end{pmatrix} \begin{array}{l} \} \quad E_1 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad E_2 \end{array}$$

We are hence able to decimate the rows of $\phi(W_a) = \mathbb{C}^{3\times 3}$ into two sets of rows by projecting W_a into two different left $\mathbb{C}S_k$-isotypic spaces. **Because**

there is one each of N_1 and N_2 in M (N_1, N_2 are multiplicity free in M), we are able to achieve the decimation of $W_a \cong \mathbb{C}^{3\times3}$ into two sets of rows corresponding to distinct left $\mathbb{C}S_k$-irreducibles by

1. **first projecting $\mathbb{C}S_n$ into W_a, a left $\mathbb{C}S_n$-isotypic space in $\mathbb{C}S_n$, and**
2. **projecting the result into the left $\mathbb{C}S_k$-isotypic spaces in $\mathbb{C}S_n$.**

However, the decomposition of M into distinct $\mathbb{C}S_k$-irreducibles by isotypic projection is impossible if N_1 and N_2 are not multiplicity free in M. Suppose, for example, that W_b is a $\mathbb{C}S_n$-isotypic space containing a left $\mathbb{C}S_n$-irreducible M' of dimension 5, and

$$M' = N_1 \oplus N_2 \oplus N_2'$$

where $N_2 \cong N_2'$ (Fig. 2.3).

$$\begin{pmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{pmatrix} \quad \begin{array}{l} \} \, N_1 \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} N_2 \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} N_2' \end{array}$$

Figure 2.3: $\phi(M')$.

Since there are three $\mathbb{C}S_k$-irreducibles in M', we wish to decimate $\phi(W_b) = \mathbb{C}^{5\times5}$ into 3 sets of rows: one set corresponding to N_1, one set corresponding to N_2, and one set corresponding to N_2'. However, because N_2 and N_2' are in the same left $\mathbb{C}S_k$-isotypic space, projection onto the left $\mathbb{C}S_n$-isotypic space W_b followed by projection onto left $\mathbb{C}S_k$-isotypic spaces cannot seperate the rows corresponding to N_2 from the rows corresponding to N_2' (Fig 2.4).

$$\left(\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array}\right) \begin{array}{l} \left.\vphantom{\circ}\right\} \ \mathbb{C}S_k\text{-isotypic containing } N_1 \\ \left.\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}\right\} \ \mathbb{C}S_k\text{-isotypic containing } N_2 \text{ and } N_2' \end{array}$$

Figure 2.4: $\phi(W_b)$.

Therefore, multiplicity can be a problem if we want to use isotypic projection by idempotents as a tool to decimate the algebra. Fortunately, **any decomposition of a $\mathbb{C}S_n$-irreducible into $\mathbb{C}S_{n-1}$ irreducibles is multiplicity free** (Clausen and Baum, 1993).

The tree in Figure 2.5 is called the Bratteli diagram for S_4, and it shows the decomposition of $\mathbb{C}S_i$-irreducible representations for $i = 1, 2, 3, 4$. Each figure (a Young diagram) on the ith level of the tree represents an isomorphism type of left/right $\mathbb{C}S_i$-irreducible module. An arrow from a diagram to a diagram below indicates that the irreducible module represented by the upper diagram is in the decomposition of the irreducible module represented by the lower diagram. Moreover, for any diagram v, the dimension of the irreducible that is represented by v is the sum of the dimensions of the irreducibles representing the diagrams that have arrows going into v. Note that a diagram on the ith level represents a partition of the number i. The Brattelli diagram implies that each isomorphism type of left/right S_i-irreducible modules (either left or right) corresponds to a partition of i. When v is a partition of i, we write $v \vdash i$. We will therefore use a partition of n alternatively to mean a $\mathbb{C}S_n$-isotypic space. In particular, we will use $a \vdash n_{right}$ to denote a right $\mathbb{C}S_n$-isotypic space, $a \vdash n_{left}$ to denote a left $\mathbb{C}S_n$-isotypic space.

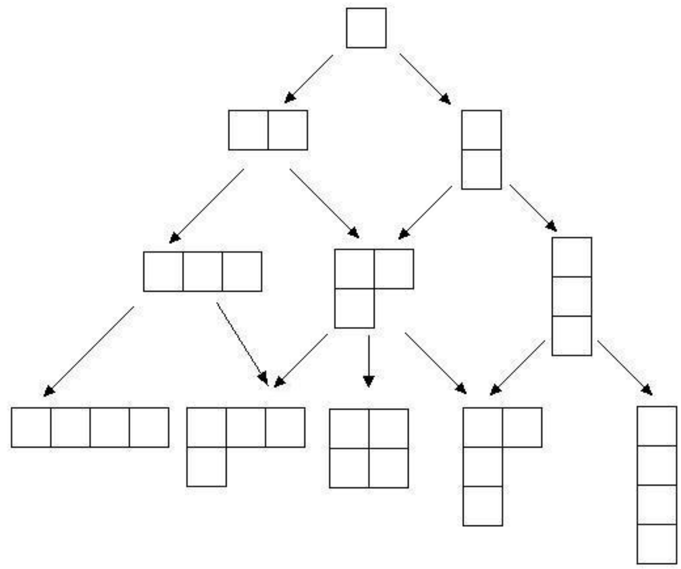

Figure 2.5: Bratteli Diagram

As shown in the Bratteli diagram, there are two nonisomorphic 1-dimensional $\mathbb{C}S_2$-irreducibles. Therefore the projection of $\mathbb{C}S_n$ into two left $\mathbb{C}S_2$-isotypic spaces can decimate the rows of $\bigoplus_i \mathbb{C}^{d_i \times d_i}$ into two sets of rows. A path from the root (the partition of 1) to the first partition of 2 corresponds to the projection onto the first set of rows, and the path from the root to the second partition of 2 corresponds to the projection onto the second set of rows. This correspondence can be extended. We can also verify the following facts:

- **Every distinct path of the same length that originates from a given Young diagram corresponds to a distinct set of rows.**
- **If** $\{a_1 \vdash i, a_2 \vdash (i+1), \ldots\}$ **are the diagrams on the path, the set of rows correponding to the path represents the space obtained by projecting** $\mathbb{C}S_n$ **into left isotypic spaces corresponding to the diagrams on the path** For example, in figure 2.6, the bold-faced path represents projection onto the first left $\mathbb{C}S_2$- isotypic space followed by projection onto the second left $\mathbb{C}S_3$-isotypic space and projection onto the second left $\mathbb{C}S_4$- isotypic space in $\mathbb{C}S_4$).

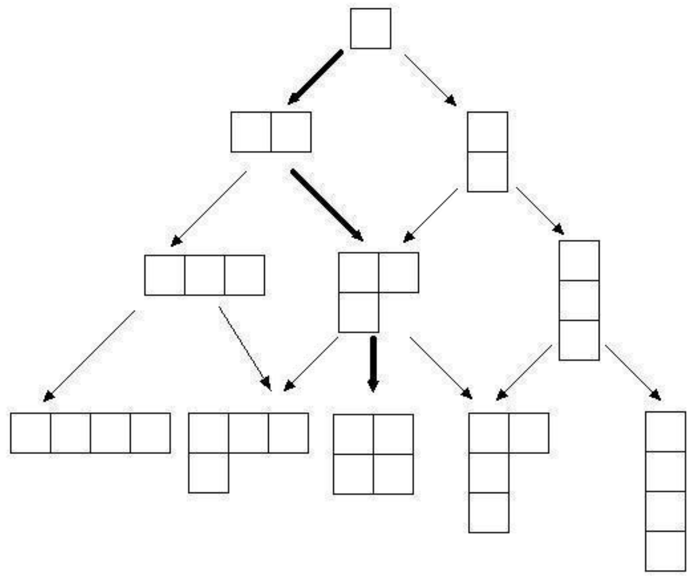

Figure 2.6: A path in the Bratteli diagram

- **The dimension of the left-irreducible module corresponding to a diagram is the number of paths from the root to the diagram.**

Therefore we can distinguish every row of the $\bigoplus_i \mathbb{C}^{d_i \times d_i} \cong \mathbb{C}S_n$ by the paths from the root to the diagrams at the nth level. This implies the following significant fact.

> **The rows of $\bigoplus_i \mathbb{C}^{d_i \times d_i} \cong \mathbb{C}S_n$ can be completely decimated (into single rows) by the composition of projections into all the left $\mathbb{C}S_i$-isotypic spaces for** $i = 1, 2, \ldots n$.

Analogously, the same thing can be said about the **columns** of $\bigoplus_i \mathbb{C}^{d_i \times d_i} \cong \mathbb{C}S_n$.

> **The columns of $\bigoplus_i \mathbb{C}^{d_i \times d_i} \cong \mathbb{C}S_n$ can be completely decimated (into single columns) by the composition of projections into all the right $\mathbb{C}S_i$-isotypic spaces for** $i = 1, 2, \ldots n$.

Together,

> **The entries of $\bigoplus_i \mathbb{C}^{d_i \times d_i} \cong \mathbb{C}S_n$ can be completely decimated (into single entries) by the composition of projections into all the left $\mathbb{C}S_i$-isotypic spaces for $i = 1, 2, \dots n$ together with the projections into all the right $\mathbb{C}S_i$-isotypic spaces for $i = 1, 2, \dots n$.**

Centrally primitive idempotents, which project the elements of $\mathbb{C}S_n$ to the corresponding isotypic spaces, therefore *seperate* the entries in $\bigoplus_i \mathbb{C}^{d_i \times d_i} \cong \mathbb{C}S_n$. For this reason, if T_m is the set of all centrally primitive idempotents of $\mathbb{C}S_m$, $\bigcup_{m=1}^{n} T_m$ is called a *separating set.*[1] The sets of entries in $\bigoplus_i \mathbb{C}^{d_i \times d_i} \cong \mathbb{C}S_n$ that separate from each other upon the projections are called **frequencies,** or frequency spaces. For example, in Figure 2.2, R_1 and R_2 are frequencies. This gives rise to the name **decimation-in-frequency (DIF).** The DIF algorithm that will be described in this chapter was pioneered by Michael Orrison. Therefore we will call this algorithm **Orrison-DIF (ODIF).**

The frequencies in the ODIF are a generalization of the frequencies f in the DIF mentioned in (1.1). Consider $\mathbb{C}C_n$, where C_n is a cyclic group generated by x. By Wedderburn's Theorem,

$$\mathbb{C}C_n \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{\text{n summands}},$$

The representation $D_f : C_n \mapsto \mathbb{C}$ in the fth $\mathbb{C}$ (counting from 0) is defined by $D_f(x) = e^{2i\pi f/n}$. Let $m|n$. Clearly, C_m is a subgroup of C_n generated by $x^{n/m}$. Suppose $f = f_1 + f_2 m$, then note that

$$\begin{aligned}
D_f(x^{n/m}) &= \left(e^{2i\pi(f_1+f_2 m)/n}\right)^{n/m} \\
&= \left(e^{2i\pi(f_1+f_2 m)}\right)^{1/m} \\
&= e^{2i\pi f_1/m} e^{2i\pi f_2} \\
&= e^{2i\pi f_1/m}
\end{aligned}$$

Thus, if $f \equiv_m f'$, the fth $\mathbb{C}$ and the f'th $\mathbb{C}$ are in same $\mathbb{C}C_m$-isotypic space, and are hence in the same frequency. Thus the classic DIF as presented in Gentleman and Sande (1966) also breaks down the DFT by the series of projection onto isotypic spaces.

[1] There are other separating sets; one of the well-known separting sets is the Jucy-Murphy elements (Malm, 2005). However, the FFT in this research uses primitive idempotents.

2.2 The ODIF Algorithm

Indeed, one may decimate $\mathbb{C}S_n$ into $n!$ frequencies by carrying out the series of projections in any order. However, order is important in facilitating fast implementation of the ODIF. In particular, our ODIF is roughly implemented as follows.

The ODIF Algorithm

Step 1: Define $\mathbb{C}S_n$ as a single frequency.

Step i $(i = 2, \ldots, (n-1))$**:**

> **Task 1:** Project the frequencies at the end of $(i-1)$th step into the right $\mathbb{C}S_i$-isotypic spaces. Define the resulting nonzero spaces as the new frequencies.
>
> **Task 2:** Project the frequencies obtained in the task 1 into left $\mathbb{C}S_i$-isotypic spaces. Define the resulting nonzero spaces as the new frequencies

Step n**:** *Project the frequencies at the end of* $(n-1)$*th step into the right* $\mathbb{C}S_n$*-isotypic spaces.*

Because we project the space into both left and right isotypic spaces, we say that the algorithm is double-sided. Note that we do not need to project into both left and right $\mathbb{C}S_n$-isotypic spaces, since any left $\mathbb{C}S_n$-isotypic space is also a right $\mathbb{C}S_n$-isotypic space in $\mathbb{C}S_n$. When either Task 1 or Task 2 is omitted from the algorithm, we call the algorithm *one sided ODIF* (OS-ODIF). Otherwise, we call the algorithm simply ODIF. As mentioned previously, projections into right (left) isotypic spaces may only decimate the $\mathbb{C}S_n$ space into single columns (rows), and there are an infinite number of choices of basis for a given column (row) when the column (row) has more than one entry. Hence, the Fourier basis resulting from the OS-ODIF, won't be unique unless each irreducible representation in the algebra is one dimensional. Here, we will focus our attention on the regular ODIF.

Notice that, after the 1st task of the ith step in the implementation of the ODIF, $\mathbb{C}S_n$ is projected into spaces that consist of intersections of right $\mathbb{C}S_i$-isotypic spaces and left $\mathbb{C}S_{i-1}$-isotypic spaces. On the other hand, after the 2nd task of the ith loop in the implementation of the ODIF, $\mathbb{C}S_n$ is projected into spaces that consist of intersections of right $\mathbb{C}S_i$-isotypic spaces and left $\mathbb{C}S_i$-isotypic spaces. This intersection is clearly a $(\mathbb{C}S_i, \mathbb{C}S_i)$-bimodule. In particular, suppose that $W_k \vdash k_{left}$ is a left $\mathbb{C}S_k$-isotypic space in $\mathbb{C}S_n$ containing a left $\mathbb{C}S_k$-irreducible V_k, and $W_h \vdash h_{right}$ is a right $\mathbb{C}S_h$-isotypic space in $\mathbb{C}S_n$ containing a right $\mathbb{C}S_h$ irreducible V_h. Then $W_h \cap W_k$ is a collection of all the $(\mathbb{C}S_k, \mathbb{C}S_h)$ bimodule irreducibles isomorphic to $V_k \otimes_{\mathbb{C}} V_h$ (Malm, 2005). Thus, we can rightfully call $W_h \cap W_k$ a $(\mathbb{C}S_k, \mathbb{C}S_h)$-(bi)isotypic space. **At each step in the loop, we are projecting the frequencies in the previous step into $(\mathbb{C}S_k, \mathbb{C}S_h)$-isotypic spaces**. The entries corresponding to the $(\mathbb{C}S_k, \mathbb{C}S_h)$-isotypic space containing $V_k \otimes_{\mathbb{C}} V_h$ have row indices corresponding to $W_k \vdash k$ and column indices corresponding to $W_h \vdash h$. Therefore each $(\mathbb{C}S_k, \mathbb{C}S_h)$-isotypic space corresponds to a unique ordered pair of partitions (W_k, W_h), and each frequency at each step corresponds to a unique ordered pair of paths in the Bratteli diagram. After the projections of previous frequencies into bi-isotypic spaces, we extract a new basis. The series of the change of basis matrices yields a factorization of the DFT matrix.

Also, if f_1 and f_2 are two different frequencies contained in $(\mathbb{C}S_{i-1}, \mathbb{C}S_i)$-isotypic space, consider the set of vectors in the next intermediate basis contained in $f_1 \cup f_2$, which result from the projection of f_1 and f_2 into $(\mathbb{C}S_i, \mathbb{C}S_i)$-isotypic spaces. Each vector in this set will lie in either f_1 or f_2, but not both. **In other words, the ODIF carries out a local change of basis in each frequency contained in $(\mathbb{C}S_{i-1}, \mathbb{C}S_i)$-isotypic spaces** so that the basis of the next set of frequencies contained in $(\mathbb{C}S_{i-1}, \mathbb{C}S_i)$-isotypic spaces respects $\mathbb{C}S_n$'s decomposition into $(\mathbb{C}S_i, \mathbb{C}S_i)$-isotypic spaces. However, the ODIF implements the change of basis inside each frequency by breaking the frequency into even smaller spaces and carrying out a local change of basis in each of them. Next, we will introduce the concept of the decimation of frequency by double coset spaces.

Decimation of Frequency by Double Coset Spaces

As stated previously, the projection of $\mathbb{C}S_n$ into a left (right) $\mathbb{C}S_k$-isotypic space can be achieved by multiplying each element in $\mathbb{C}S_n$ on the left (right) by the centrally primitive idempotent of $\mathbb{C}S_k$ corresponding to the associated partition of k. If $a \vdash k$, denote the centrally primitive idempotent corresponding to this partition by $e_{a \vdash k}$. Then each frequency in a given step of the ODIF algorithm can be considered as

$$\Big(\prod_{i=1}^{k} e_{a^{(i)}}\Big) \mathbb{C}S_n \Big(\prod_{j=1}^{h} e_{b^{(h+1-j)}}\Big)$$

for some k, h, where $\{a^{(x)}\}_{x=1}^{k}$ and $\{b^{(x)}\}_{x=1}^{h}$ are unique sequences of partitions such that $a^{(x)}, b^{(x)} \vdash x$. Here, we take advantage of the expression of the centrally primitive idempotents (Dummit and Foote, 1991). Because $e_{a \vdash k} \in \mathbb{C}S_k$, each vector in the spanning set of each frequency obtained from projections by centrally primitive idempotents(i.e, if S is the standard basis,) is contained in a particular (S_k, S_h) double coset space $\mathbb{C}(S_k g S_h)$. Therefore, at the each step of ODIF, we may sort the basis of a given frequency by the double coset spaces to which they belong. In other words, we can decimate the frequency by the double cosets.

Suppose, for example, that f is a frequency contained in a $(\mathbb{C}S_{k-1}, \mathbb{C}S_k)$-isotypic space and it decimates into $f_1, f_2, \ldots f_\ell$, where f_i and f_j belong respectively to distinct (S_{k-1}, S_k) double coset spaces A_i and A_j. Then it is clear that $e_{a \vdash k} f_i$ and $e_{a \vdash k} f_j$ can have nontrivial intersection if and only if A_i and A_j are in the same (S_k, S_k) double coset space. Thus, we should give a name to the union of all f_is that belong to the same (S_k, S_k) double coset space.

A double coset frequency (DCF) contained in a $(\mathbb{C}S_{k-1}, \mathbb{C}S_k)$ isotypic space is an intersection of a frequency in the $(\mathbb{C}S_{k-1}, \mathbb{C}S_k)$ isotypic space and an (S_k, S_k) double coset space. Frequencies at each step therefore decomposes into a direct sum of DCFs. In terms of signal processing, a DCF can be rightfully called a **frequency decimated by time.**

Thus, each step in the ODIF is not only the collection of a local change of basis within each frequency, but also the collection of a change of basis within each DCF. Indeed, the runtime of the ODIF is largely determined by the size of each DCF at each step; specifically, the change of basis at each step of the ODIF will be the direct sum of the change of bases in each DCF. In the next chapter, we will discuss this matter in further detail.

Notice that the only property that the ODIF used was multiplicity freeness of the restriction diagram of S_n's irreducibles (Bratteli diagram). Therefore, this method works for all $\mathbb{C}G$ for which its restriction diagram is multiplicity free. The following example is for $G = C_4$, which satisfies this property. C_4 is also abelian, hence the role of double cosets are played by one sided cosets.

2.2.1 Example: Abelian Case

Consider $\mathbb{C}C_4$. We implement the ODIF on this algebra.

Step 1 The original basis of $\mathbb{C}C_4$ is $\{1, x^2, x, x^3\}$. The entire space is a $\mathbb{C}C_1$-isotypic space; hence the entire space is one frequency. We decimate this frequency into DCFs by C_2-coset spaces. Denote the jth DCF in the ith step by f_{ij}. Then

$$f_{11} = span\{1, x^2\}, f_{12} = span\{x, x^3\}.$$

Let e_{ij} denote the ith centrally primitive idempotent of $\mathbb{C}C_j$. From the formula given in Dummit and Foote (1991) we can easily compute these idempotents: $e_{12} = \frac{1}{2}(1 + x^2)$ and $e_{22} = \frac{1}{2}(1 - x^2)$. We project each DCFs into the $\mathbb{C}C_2$ isotypic spaces using these elements. Then

$$\begin{aligned}
e_{12}f_{11} =& span\{e_{12}1, e_{11}x^2\} = span\{e_{12}, e_{12}\} &&= span\{e_{11}\} \\
e_{22}f_{11} =& span\{e_{22}1, e_{12}x^2\} = span\{e_{22}, -e_{22}\} &&= span\{e_{12}\} \\
e_{12}f_{12} =& span\{e_{12}x, e_{11}x^3\} = span\{e_{12}x, e_{12}x\} &&= span\{e_{11}x\} \\
e_{22}f_{12} =& span\{e_{22}x, e_{12}x^3\} = span\{e_{22}x, -e_{22}x\} &&= span\{e_{12}x\}.
\end{aligned}$$

The intermediate basis at this step is hence $\{e_{11}, e_{12}, e_{11}x, e_{12}x\}$, and the new frequencies are $e_{11}f_{11} \cup e_{11}f_{12}$ and $e_{12}f_{11} \cup e_{12}f_{12}$. In vector form, the elements of the intermediate basis are (in the same order)

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Note that, the choice of the basis for each projection of the DCF was very easy because $e_{1j}1$ is a multiple of $e_{1j}x^2$ and $e_{1j}x$ is a multiple of $e_{1j}x^3$ for all j. The reason for this easy choice is simple: it is because the multiplicity of $\mathbb{C}C_2$-irreducible modules in each coset space is 1, and each coset contains only one irreducible corresponding to e_{1j} for both j. This renders each projection of the DCF one dimensional.

The matrix of the change of basis from the original basis to the new basis is:

$$\begin{pmatrix} 1/2 & & 1/2 & \\ 1/2 & & -1/2 & \\ & 1/2 & & 1/2 \\ & 1/2 & & -1/2 \end{pmatrix}.$$

Because there are two DCFs (f_{11} and f_{12}) of dimension 2, we see two blocks of dimension 2×2 in this change of basis matrix. Each block represents a change of basis within a DCF.

Step 2 The frequencies at this step are $e_{11}f_{11} \cup e_{11}f_{12}$ and $e_{12}f_{11} \cup e_{12}f_{12}$. Because the two C_2 coset spaces are both contained in the same (unique)

C_4 coset space, these frequencies are also DCFs at this step. Let the first be f_{21} and the second be f_{22}. Also, $e_{14} = \frac{1}{4}(1 + x + x^2 + x^3)$, $e_{24} = \frac{1}{4}(1 - ix - x^2 + ix^3)$, $e_{34} = \frac{1}{4}(1 - x + x^2 - x^3)$, $e_{44} = \frac{1}{4}(1 + ix - x^2 - ix^3)$. We project each DCF to each of the four $\mathbb{C}C_4$-isotypic spaces corresponding to these idempotents. Then

$$\begin{aligned}
e_{14}f_{21} &= span\{e_{14}e_{11}, e_{14}e_{11}x\} = span\{2 * e_{14}, 2 * e_{14}\} = span\{e_{14}\}.\\
e_{24}f_{21} &= span\{e_{24}e_{11}, e_{24}e_{11}x\} = span\{0, 0\} = 0.\\
e_{34}f_{21} &= span\{e_{34}e_{11}, e_{34}e_{11}x\} = span\{2 * e_{34}, -2 * e_{34}\} = span\{e_{34}\}.\\
e_{44}f_{22} &= span\{e_{44}e_{11}, e_{44}e_{11}x\} = span\{0, 0\} = 0.\\
e_{14}f_{21} &= span\{e_{14}e_{12}, e_{14}e_{12}x\} = span\{0, 0\} = 0.\\
e_{24}f_{21} &= span\{e_{24}e_{12}, e_{24}e_{12}x\} = span\{2 * e_{24}, 2i * e_{24}\} = span\{e_{24}\}.\\
e_{34}f_{21} &= span\{e_{34}e_{12}, e_{34}e_{12}x\} = span\{0, 0\} = 0\\
e_{44}f_{22} &= span\{e_{44}e_{12}, e_{44}e_{12}x\} = span\{2 * e_{44}, -2i * e_{44}\} = span\{e_{44}\}.
\end{aligned}$$

The last basis (the Fourier basis) is hence $e_{14}, e_{24}, e_{34}, e_{44}$.[2] In vector form, these elements are

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}.$$

The matrix of the change of basis from the second basis to this basis is

$$\begin{pmatrix} 2 & & 2 & \\ & 2 & & 2i \\ 2 & & -2 & \\ & 2 & & -2i \end{pmatrix}.$$

Again, because there are two DCFs (f_{21} and f_{22}) of dimension 2, we see two blocks of dimension 2×2 in this change of basis matrix. Finally, we have the factorization of the DFT matrix of $\mathbb{C}C_4$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2i \\ 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2i \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & -1/2 \end{pmatrix}$$

[2]Because any $\mathbb{C}C_4$-isotypic space is 1-dimensional, this Fourier basis for $\mathbb{C}C_4$ is unique. In particular, for any abelian G, any $\mathbb{C}G$-isotypic space is 1-dimensional, and the Fourier basis of $\mathbb{C}G$ is unique.

This factorization is equivalent to the one that can be obtained by Gentleman-Sande-DIF algorithm (James and Kerber, 1981).

2.2.2 Example: Non-Abelian Case

Consider $\mathbb{C}S_3$. From Wedderburn's Theorem,

$$\mathbb{C}S_3 = I_1 \oplus I_2 \oplus I_3 \cong \mathbb{C}^{1\times 1} \oplus \mathbb{C}^{2\times 2} \oplus \mathbb{C}^{1\times 1},$$

where the I_is are left $\mathbb{C}S_3$-isotypic spaces. Further, $I_1 = L_1$, $I_2 = L_2 \oplus L_2' \cong L_2 \oplus L_2$, and $I_3 \cong L_3$ where the L_is are left $\mathbb{C}S_3$-irreducibles. From the Bratteli diagram (Fig 2.5), we can see that I_1 corresponds to the partition (3) of 3, I_2 corresponds to the partition $(2,1)$ of 3, and I_3 corresponds to the partition $(1,1,1)$ of 3.

Under Wedderburn's isomorphism ϕ with the doubly adapted basis mentioned in 2.1.1,

$$\mathbb{C}S_3 \cong \begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & & & \bullet \end{pmatrix}$$

$$L_1 \cong \begin{pmatrix} \bullet & & & \\ & 0 & 0 & \\ & 0 & 0 & \\ & & & 0 \end{pmatrix} \quad L_2 \cong \begin{pmatrix} 0 & & & \\ & \bullet & 0 & \\ & \bullet & 0 & \\ & & & 0 \end{pmatrix}$$

$$L_3 \cong \begin{pmatrix} 0 & & & \\ & 0 & \bullet & \\ & 0 & \bullet & \\ & & & 0 \end{pmatrix} \quad L_4 \cong \begin{pmatrix} 0 & & & \\ & 0 & 0 & \\ & 0 & 0 & \\ & & & \bullet \end{pmatrix}. \tag{2.1}$$

Step 1 The standard basis for $\mathbb{C}S_3$ is $\{(1), (12), (23), (123), (132), (13)\}$. The entire space is a $\mathbb{C}S_1$-isotypic space; hence the entire space is one frequency. Again, denote the jth DCF in the ith step by f_{ij}. We decimate this frequency into three DCFs by (S_2, S_1) double cosets:

$$f_{11} = span\{(1), (12)\}, \quad f_{12} = span\{(23), (123)\}, \quad f_{13} = span\{(132), (13)\}.$$

From the formula given in Dummit and Foote (1991), $e_{(2)\vdash 2} = \frac{1}{2}(1 + (12))$ and $e_{(1,1)\vdash 2} = \frac{1}{2}(1 - (12))$. We project each DCF into $(\mathbb{C}S_2, \mathbb{C}S_1)$-isotypic spaces:

$$\begin{aligned}
e_{(2)}f_{11} &= span\{e_{(2)}(1), e_{(2)}(12)\} && = span\{e_{(2)}(1)\}\\
e_{(2)}f_{12} &= span\{e_{(2)}(23), e_{(2)}(123)\} && = span\{e_{(2)}(23)\}\\
e_{(2)}f_{13} &= span\{e_{(2)}(132), e_{(2)}(13)\} && = span\{e_{(2)}(132)\}\\
e_{(1,1)}f_{11} &= span\{e_{(1,1)}(1), e_{(1,1)}(12)\} && = span\{e_{(1,1)}(1)\}\\
e_{(1,1)}f_{12} &= span\{e_{(1,1)}(23), e_{(1,1)}(123)\} && = span\{e_{(1,1)}(23)\}\\
e_{(1,1)}f_{13} &= span\{e_{(1,1)}(132), e_{(1,1)}(13)\} && = span\{e_{(1,1)}(132)\}.
\end{aligned}$$

The intermediate basis at this step is hence

$$\{e_{(2)}(1), e_{(2)}(23), e_{(2)}(132), e_{(1,1)}(1), e_{(1,1)}(23), e_{(1,1)}(132)\}.$$

In vector form, these bases are

$$\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ -1/2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

The matrix of the change of basis from the original basis to this basis is

$$\begin{pmatrix} 1 & 1 & & & & \\ 1 & -1 & & & & \\ & & 1 & 1 & & \\ & & 1 & -1 & & \\ & & & & 1 & 1 \\ & & & & 1 & -1 \end{pmatrix}.$$

Because there are three DCFs (f_{11}, f_{12}, f_{13}) of dimension 2, we see three blocks of dimension 2×2 in this change of basis matrix.

By looking at the idempotents appearing in the basis vectors, we see that

- $e_{(2)}f_{11} \cup e_{(2)}f_{12} \cup e_{(2)}f_{13}$ is the frequency space in the next step corresponding to the ordered pair of partitions $\{(2), (1)\}$.
- $e_{(1,1)}f_{11} \cup e_{(1,1)}f_{12} \cup e_{(1,1)}f_{13}$ is the frequency space in the next step corresponding to the ordered pair of partitions $\{(1,1), (1)\}$.

Without loss of generality, the first and the second row in (2.1) correspond to the frequency $e_{(2)}\mathbb{C}S_3$. The third and the fourth row correspond to the frequency $e_{(1,1)}\mathbb{C}S_3$:

$$e_{(2)}\mathbb{C}S_3 = \begin{pmatrix} \bullet & & & \\ & \bullet & \bullet & \\ & 0 & 0 & \\ & & & 0 \end{pmatrix} \qquad e_{(1,1)}\mathbb{C}S_3 = \begin{pmatrix} 0 & & & \\ & 0 & 0 & \\ & \bullet & \bullet & \\ & & & \bullet \end{pmatrix}.$$

Step 2 The frequencies at this step are

$$e_{(2)}f_{11} \cup e_{(2)}f_{12} \cup e_{(2)}f_{13} \qquad \text{and} \qquad e_{(1,1)}f_{11} \cup e_{(1,1)}f_{12} \cup e_{(1,1)}f_{13}.$$

We decimate each frequency by (S_2, S_2) double cosets. The two (S_2, S_2) double cosets in S_3 are

$$S_2(1)S_2 = \{(1), (12)\} \qquad \text{and} \qquad S_2(23)S_2 = \{(23), (123), (132), (13)\}.$$

Therefore we can see from the basis in the step 1 that the DCFs at this step are:

$$\begin{aligned}
f_{21} &= e_{(2)}f_{11} && = span\{e_{(2)}(1)\} \\
f_{22} &= e_{(2)}f_{12} \cup e_{(2)}f_{13} && = span\{e_{(2)}(23), e_{(2)}(132)\} \\
f_{23} &= e_{(1,1)}f_{11} && = span\{e_{(1,1)}(1)\} \\
f_{24} &= e_{(1,1)}f_{12} \cup e_{(2)}f_{13} && = span\{e_{(1,1)}(23), e_{(1,1)}(132)\}.
\end{aligned}$$

We project each *DCF* to an (S_2, S_2)-isotypic space. Each projection will be

$$
\begin{aligned}
f_{21}e_{(2)} &= span\{e_{(2)}(1)e_{(2)}\} && \\
f_{21}e_{(1,1)} &= span\{e_{(2)}(1)e_{(1,1)}\} && = span\{0\} = 0 \\[1ex]
f_{22}e_{(2)} &= span\{e_{(2)}(23)e_{(2)}, e_{(2)}(132)e_{(2)}\} && = span\{e_{(2)}(23)e_{(2)}\} \\
f_{22}e_{(1,1)} &= span\{e_{(2)}(23)e_{(1,1)}, e_{(2)}(132)e_{(1,1)}\} && = span\{e_{(2)}(23)e_{(1,1)}\} \\[1ex]
f_{23}e_{(2)} &= span\{e_{(1,1)}(1)e_{(2)}\} && = span\{0\} = 0 \\
f_{23}e_{(1,1)} &= span\{e_{(1,1)}(1)e_{(1,1)}\} && \\[1ex]
f_{24}e_{(2)} &= span\{e_{(1,1)}(23)e_{(2)}, e_{(1,1)}(132)e_{(2)}\} && = span\{e_{(1,1)}(23)e_{(2)}\} \\
f_{24}e_{(1,1)} &= span\{e_{(1,1)}(23)e_{(1,1)}, e_{(1,1)}(132)e_{(1,1)}\} && = span\{e_{(1,1)}(23)e_{(1,1)}\}.
\end{aligned}
$$

The intermediate basis of this step is hence

$$\{e_{(2)}(1)e_{(2)}, e_{(1,1)}(1)e_{(1,1)}, e_{(2)}(23)e_{(2)}, e_{(2)}(23)e_{(1,1)}, e_{(1,1)}(23)e_{(2)}, e_{(1,1)}(23)e_{(1.1)}\}.$$

In vector form, the bases are

$$
\begin{pmatrix} 1/4 \\ 1/4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 1/4 \\ -1/4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 1/4 \\ 1/4 \\ -1/4 \\ -1/4 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 1/4 \\ -1/4 \\ 1/4 \\ -1/4 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 1/4 \\ -1/4 \\ -1/4 \\ 1/4 \end{pmatrix}.
$$

The matrix of the change of basis from the second basis to this new basis is

$$
\begin{pmatrix}
1/4 & & & & & \\
& 1/4 & & & & \\
& & 1/4 & & 1/4 & \\
& & 1/4 & & -1/4 & \\
& & & 1/4 & & 1/4 \\
& & & 1/4 & & -1/4
\end{pmatrix}.
$$

Because there are two DCFs of dimension 1 (f_{21}, f_{23}) and two DCFs of dimension 2 (f_{22}, f_{24}), we see two blocks of dimension 1×1 and two blocks of dimension 2×2 in this change of basis matrix.

$f_{21}e_{(1,1)} = 0$ and $f_{23}e_{(2)} = 0$ imply that $S_2(1)S_2$ does not contain any irreducible corresponding to the ordered pair of partitions $\{(2), (1,1)\}$ nor

$\{(1,1),(2)\}$. By looking at the idempotents appearing in the basis vectors, we see that

- $f_{22}e_{(1,1)}$ is the frequency space corresponding to the ordered pair of partitions $\{(2),(1,1)\}$.

- $f_{21}e_{(2)} \cup f_{22}e_{(2)}$ is the frequency space corresponding to the ordered pair of partitions $\{(2),(2)\}$.

- $f_{23}e_{(1,1)} \cup f_{24}e_{(1,1)}$ is the frequency space corresponding to the ordered pair of partitions $\{(1,1),(1,1)\}$.

- $f_{24}e_{(2)}$ is the frequency space corresponding to the ordered pair of partitions $\{(1,1),(2)\}$.

In the frequency domain, each one of them looks like the following:

$$f_{21}e_{(2)} \cup f_{22}e_{(2)} \cong \begin{pmatrix} \bullet & & & \\ & \bullet & 0 & \\ & 0 & 0 & \\ & & & 0 \end{pmatrix} \qquad f_{22}e_{(1,1)} \cong \begin{pmatrix} 0 & & & \\ & 0 & \bullet & \\ & 0 & 0 & \\ & & & 0 \end{pmatrix}$$

$$f_{23}e_{(1,1)} \cup f_{24}e_{(1,1)} \cong \begin{pmatrix} 0 & & & \\ & 0 & 0 & \\ & 0 & \bullet & \\ & & & \bullet \end{pmatrix} \qquad f_{24}e_{(2)} \cong \begin{pmatrix} 0 & & & \\ & 0 & 0 & \\ & \bullet & 0 & \\ & & & 0 \end{pmatrix}.$$

Step 3 Note that there is only one (S_2, S_3) double coset in S_3, which is S_3 itself. Therefore the four frequencies

$$f_{22}e_{(1,1)}, \quad f_{21}e_{(2)} \cup f_{22}e_{(2)}, \quad f_{23}e_{(1,1)} \cup f_{24}e_{(1,1)}, \quad f_{24}e_{(2)}$$

are DCFs themselves. Hence let

$$f_{31} = f_{22}e_{(1,1)}, \quad f_{32} = f_{21}e_{(2)} \cup f_{22}e_{(2)}, \quad f_{33} = f_{23}e_{(1,1)} \cup f_{24}e_{(1,1)}, \quad f_{34} = f_{24}e_{(2)}.$$

We project each one of them into $(\mathbb{C}S_2, \mathbb{C}S_3)$- isotypic spaces. The three idempotents of $\mathbb{C}S_3$ correspond to the three partitions of 3; namely,

$$\begin{aligned}
e_{(3)} =& \frac{1}{6}((1)+(23)+(12)+(123)+(132)+(13)) \\
e_{(2,1)} =& \frac{1}{3}(2(1)-(123)-(132)) \\
e_{(1,1,1)} =& \frac{1}{6}((1)-(23)-(12)+(123)+(132)-(13)).
\end{aligned}$$

Each projection will be

$$\begin{aligned}
e_{(3)}f_{31} &= span\{0\} && = 0 \\
e_{(2,1)}f_{31} &= span\left\{\frac{1}{4}((23)+(123)-(132)-(13)), \frac{1}{4}(-(23)-(123)+(132)+(13))\right\} \\
e_{(1,1,1)}f_{31} &= span\{0\} && = 0 \\
\\
e_{(3)}f_{32} &= span\left\{\frac{1}{6}((1)+(23)+(12)+(123)+(132)+(13))\right\} \\
e_{(2,1)}f_{32} &= span\left\{\frac{1}{12}(-2(1)+(23)-2(12)+(123)+(132)+(13))\right\} \\
e_{(1,1,1)}f_{32} &= span\{0\} && = 0 \\
\\
e_{(3)}f_{33} &= span\{0\} && = 0 \\
e_{(2,1)}f_{33} &= span\left\{\frac{1}{12}(2(1)+(23)-2(12)-(123)-(132)+(13))\right\} \\
e_{(1,1,1)}f_{33} &= span\left\{\frac{1}{6}((1)-(23)-(12)+(123)+(132)-(13))\right\} \\
\\
e_{(3)}f_{34} &= span\{0\} && = 0 \\
e_{(2,1)}f_{34} &= span\left\{\frac{1}{4}((23)-(123)+(132)-(13)), \frac{1}{4}(-(23)+(123)-(132)+(13))\right\} \\
e_{(1,1,1)}f_{34} &= span\{0\} && = 0.
\end{aligned}$$

The last basis is hence

$$\begin{aligned}
&\frac{1}{6}((1)+(12)+(23)+(123)+(132)+(13)), \\
&\frac{1}{12}(-2(1)-2(12)+(23)+(123)+(132)+(13)),
\end{aligned}$$

$$\frac{1}{12}\big(2(1) - 2(12) + (23) - (123) - (132) + (13)\big),$$
$$\frac{1}{6}\big((1) - (12) - (23) + (123) + (132) - (13)\big)$$
$$\frac{1}{4}\big(-(23) - (123) + (132) + (13)\big), \qquad \frac{1}{4}\big(-(23) + (123) - (132) + (13)\big).$$

In vector form, the basis vectors are:

$$\begin{pmatrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{pmatrix}, \begin{pmatrix} -2/12 \\ -2/12 \\ 1/12 \\ 1/12 \\ 1/12 \\ 1/12 \end{pmatrix}, \begin{pmatrix} 2/12 \\ -2/12 \\ 1/12 \\ -1/12 \\ -1/12 \\ 1/12 \end{pmatrix}, \begin{pmatrix} 1/6 \\ -1/6 \\ -1/6 \\ 1/6 \\ 1/6 \\ -1/6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1/4 \\ -1/4 \\ 1/4 \\ 1/4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{pmatrix}.$$

The matrix of the change of basis from the third basis to this last basis is

$$\begin{pmatrix} 1/2 & & 1 & & & \\ -1 & & 1 & & & \\ & 1 & & & & 1 \\ & 1/2 & & & & -1 \\ & & & -1 & & \\ & & & & -1 & \end{pmatrix}.$$

Because there are two DCFs of dimension 1 (f_{31}, f_{34}) and two DCFs of dimension 2 (f_{32}, f_{33}), we see two blocks of dimension 1×1 and two blocks of dimension 2×2 in this change of basis matrix.

We will conclude this section by presenting a theorem that reveals important properties of the frequencies in the ODIF. The theorem, in particular, allows us to know the number of the frequencies at each step and their dimensions. Let $P(n)$ be the number of distinct $\mathbb{C}S_n$-isotypic spaces (or equivalently, number of partitions of n). Then the number of distinct $(\mathbb{C}S_k, \mathbb{C}S_h)$-isotypic spaces is $P(k) * P(h)$. Also, denote the jth partition of n corresponding to a left $\mathbb{C}S_n$-isotypic space by $a_j \vdash n$, and the jth partition of n corresponding to a right $\mathbb{C}S_n$-irreducible by $b_j \vdash n$. Moreover, denote the $(\mathbb{C}S_k, \mathbb{C}S_h)$-isotypic space in S_n corresponding to $a_s \vdash k_{left} \otimes_{\mathbb{C}} b_t \vdash h_{right}$ by $W_{a_s \vdash k, b_t \vdash h}$, and let $P_{a_s \vdash k, b_t \vdash h}$ be the number of irreducible $(\mathbb{C}S_h, \mathbb{C}S_k)$-bimodules in $W_{a_s \vdash k, b_t \vdash h}$. If k, h is self implied, I denote $W_{a_s \vdash k, b_t \vdash h}$ by W_{a_s, b_t}, and $P_{a_s \vdash k, b_t \vdash h}$ by P_{a_s, b_t}.

Theorem 2.1. *Suppose that we have just completed the first task of the ith loop in the algorithm. The intermediate basis at this step respects $\mathbb{C}S_n$'s unique decomposition into $(\mathbb{C}S_{i-1}, \mathbb{C}S_i)$ isotypic spaces. Let $\{V_{s,t,p} : 1 \leq s \leq P(i-1), 1 \leq t \leq P(i), 1 \leq p \leq P_{a_s \vdash (i-1), b_t \vdash i}\}$ be the set of all distinct irreducible $(\mathbb{C}S_{i-1}, \mathbb{C}S_i)$-bimodules under some decomposition of the isotypic spaces into irreducible bimodules, where $V_{s,t,p}$ and $V_{s,t,q}$ are in W_{a_s,b_t} for all p, q. Then the set of distinct frequencies at this step can be indexed by*

$$\{f_{(a_s \vdash i-1, b_t \vdash i),m} : 1 \leq s \leq P(i-1), 1 \leq t \leq P(i), 1 \leq m \leq dim(V_{s,t,p})\} \quad (2.2)$$

where $dim(f_{(a_s,b_t),m}) = P_{a_s,b_t}$. Also, $f_{(a_s,b_t),m} \cap V_{s,t,p}$ is 1-dimensional for all s, t, m, p and

$$f_{(a_s,b_t),m} = \bigoplus_{p=1}^{P_{a_s,b_t}} \left(f_{(a_s,b_t),m} \cap V_{s,t,p}\right). \quad (2.3)$$

Furthermore,

$$\bigoplus_{t=1}^{dim(V_{s,t,p})} f_{(a_s,b_t),m} = W_{a_s,b_t} \quad (2.4)$$

and, each $f_{(a_s,b_t),m}$ can be expressed as

$$W_{1,b^{(i)}} \cap \left(\bigcap_{\ell=1}^{i-1} \left(W_{a^{(\ell)},1} \cap W_{1,b^{(\ell)}}\right)\right) \quad (2.5)$$

Where $\{a^{(\ell)}\}_{\ell=1}^{i-1}$ and $\{b^{(\ell)}\}_{\ell=1}^{i}$ are unique sequences of partitions such that $a^{(\ell)}, b^{(\ell)} \vdash \ell$.

Likewise, suppose that we have just completed the second task of the ith loop in the algorithm. The intermediate basis at this step respects $\mathbb{C}S_n$'s unique decomposition into $(\mathbb{C}S_i, \mathbb{C}S_i)$ isotypic spaces. Let $\{V_{s,t,p} : 1 \leq s, t \leq P(i), 1 \leq p \leq P_{a_s \vdash i, b_t \vdash i}\}$ be the set of all distinct irreducible $(\mathbb{C}S_i, \mathbb{C}S_i)$-bimodules under some decomposition of isotypic spaces into irreducible bimodules, where $V_{s,t,p}$ and $V_{s,t,q}$ are in W_{a_s,b_t} for all p, q. Then the set of distinct frequencies at this step can be indexed by

$$\{f_{(a_s \vdash i, b_t \vdash i),m} : 1 \leq s \leq P(i), 1 \leq t \leq P(i), 1 \leq m \leq dim(V_{s,t,p})\} \quad (2.6)$$

where $dim(f_{(a_s,b_t),m}) = P_{a_s,b_t}$. Also, $f_{(a_s,b_t),m} \cap V_{s,t,p}$ is 1-dimensional for all s, t, m, p and

$$f_{(a_s,b_t),m} = \bigoplus_{p=1}^{P_{a_s,b_t}} \left(f_{(a_s,b_t),m} \cap V_{s,t,p}\right). \quad (2.7)$$

Furthermore,

$$\bigoplus_{t=1}^{dim(V_{s,t,p})} f_{(a_s,b_t),m} = W_{a_s,b_t} \tag{2.8}$$

and, each $f_{(a_s,b_t),m}$ *can be expressed as*

$$\bigcap_{\ell=1}^{i} \left(W_{a^{(\ell)},1} \cap W_{1,b^{(\ell)}}\right) \tag{2.9}$$

Where $\{a^{(\ell)}\}_{\ell=1}^{i}$ *and* $\{b^{(\ell)}\}_{\ell=1}^{i}$ *are unique sequences of partitions such that* $a^{(\ell)}, b^{(\ell)} \vdash \ell$.

Proof. The key behind this proof is the fact that the Bratteli diagram is multiplicity free. We proceed by induction. Consider the case $i = 2$. The frequency in the first task of step 2 results from projecting the standard basis to $(\mathbb{C}S_1, \mathbb{C}S_2)$ isotypic spaces. Every irreducible $(\mathbb{C}S_1, \mathbb{C}S_2)$-bimodule is one-dimensional. Thus $dim(f_{a_s \vdash 2, b_t \vdash 2,1})$ is precisely the number of irreducible $(\mathbb{C}S_1, \mathbb{C}S_2)$ bimodules in the $(\mathbb{C}S_1, \mathbb{C}S_2)$-isotypic space corresponding to (a_s, b_t), or P_{a_s,b_t}. These satisfy (2.2). (2.3), (2.4) are satisfied trivially. (2.5) follows because the entire $\mathbb{C}S_n$ is the $(\mathbb{C}S_1, \mathbb{C}S_1)$-isotypic space, and each frequency at this step is $\mathbb{C}S_n \cap W_{(1)\vdash 1, b_t \vdash 2}$ for some t. The statements from (2.6) through (2.9) follows from an argument analogous to the following inductive step.

Assume that the statements from (2.2) through (2.9) are true for $i = k - 1$, and suppose that we are about to implement the first task of the kth step. Denote the jth partition of $k-1$ by $a_j \vdash (k-1)$, and denote jth partition of k by $b_j \vdash k$. Consider the new frequencies contained in W_{a_s,b_t}. ODIF obtains this by the projection of the previous frequencies into $W_{(1)\vdash 1, b_t}$. By the inductive hypothesis, an old frequency takes the following form:

$$\bigcap_{\ell=1}^{i-1} \left(W_{a^{(\ell)},1} \cap W_{1,b^{(\ell)}}\right)$$

where $\{a^{(\ell)}\}_{\ell=1}^{i-1}$ and $\{b^{(\ell)}\}_{\ell=1}^{i-1}$ are unique sequences of partitions such that $a^{(\ell)}, b^{(\ell)} \vdash \ell$. Thus, the new frequency takes the following form:

$$W_{(1)\vdash 1, b_t} \cap \Big(\bigcap_{\ell=1}^{i-1} \left(W_{a^{(\ell)},1} \cap W_{1,b^{(\ell)}}\right)\Big).$$

So (2.5) follows. Suppose further that J' is a set of ordered pairs of elements from $\{1, \ldots, P(k-1)\}$, and that a $(\mathbb{C}S_{k-1}, \mathbb{C}S_k)$ irreducible bimodule in W_{a_s,b_t} is isomorphic as $(\mathbb{C}S_{k-1}, \mathbb{C}S_{k-1})$-module to

$$\bigoplus_{(j_1,j_2)\in J'} V_{j_1,j_2}$$

where $V_{j_1,j_2} \in W_{a_{j_1}\vdash(k-1), a_{j_2}\vdash(k-1)}$ (so $V_{j_1,j_2} \cong V_{j_1,j_2,p}$ for all $p = 1, \ldots P_{a_{j_1},a_{j_2}}$). Notice that we can assert this because the Bratteli diagram is multiplicity free.

Now, consider a specific decomposition of W_{a_s,b_t} into P_{a_s,b_t} isomorphic irreducible $(\mathbb{C}S_{k-1}, \mathbb{C}S_k)$-bimodules. We denote these irreducible bimodules by M_r, where $r = 1, \ldots, P_{a_s,b_t}$:

$$W_{a_s,b_t} = \bigoplus_{r=1}^{P_{a_s,b_t}} M_r \qquad (M_{r_1} \cong M_{r_2} \quad \forall r_1, r_2).$$

Also consider their decomposition into irreducible $(\mathbb{C}S_{k-1}, \mathbb{C}S_{k-1})$ -bimodules:

$$M_r = \bigoplus_{(j_1,j_2)\in J'} V^{(r)}_{j_1,j_2} \qquad (V^{(r)}_{j_1,j_2} \cong V_{j_1,j_2} \quad \forall r). \tag{2.10}$$

Meanwhile, by the induction hypothesis, under some decomposition

$$W_{a_{j_1},a_{j_2}} = \bigoplus_{p=1}^{P_{a_{j_1},a_{j_2}}} V_{j_1,j_2,p}$$

where, for all j_1, j_2, m, $f_{(a_{j_1},a_{j_2}),m}$'s projection onto each $V_{j_1,j_2,p}$ is 1-dimensional. Also, again by the induction hypothesis, $f_{(a_{j_1},a_{j_2}),m}$ is

$$\bigcap_{\ell=1}^{k-1} \left(W_{a^{(\ell)},1} \cap W_{1,b^{(\ell)}}\right)$$

where $\{a^{(\ell)}\}_{\ell=1}^{k-1}$ and $\{b^{(\ell)}\}_{\ell=1}^{k-1}$ are unique sequences of partitions such that $a^{(\ell)}, b^{(\ell)} \vdash k$. Note that, for any x, y, The number of irreducible $(\mathbb{C}S_x, \mathbb{C}S_y)$-bimodules contained in the decomposition of $V_{j_1,j_2,p} \cap W_{a^{(x)},b^{(y)}}$ depends only on the isomorphism type of $V_{j_1,j_2,p}$.

If η is the number of irreducible $(\mathbb{C}S_x, \mathbb{C}S_y)$-bimodules contained in the decomposition of $V_{j_1,j_2,p} \cap W_{a^{(x)},b^{(y)}}$, then $V_{j_1,j_2,p} \cap W_{a^{(x)},b^{(y)}}$ is isomorphic to

$\eta\hat{V}$ (i.e. direct sum of η copies of $\hat{V}$), where $\hat{V}$ is an irreducible $(\mathbb{C}S_x, \mathbb{C}S_y)$-bimodule. Then for any other pair of integers $\hat{x} \leq x, \hat{y} \leq y$,

$$V_{j_1,j_2,p} \cap W_{a^{(x)},b^{(y)}} \cap W_{a^{(\hat{x})},b^{(\hat{y})}} \cong \eta(\hat{V} \cap W_{a^{(\hat{x})},b^{(\hat{y})}}).$$

Thus the number of irreducible $(\mathbb{C}S_{\hat{x}}, \mathbb{C}S_{\hat{y}})$-bimodules contained in the decomposition of $V_{j_1,j_2,p} \cap W_{a^{(x)},b^{(y)}} \cap W_{a^{(\hat{x})},b^{(\hat{y})}}$ also depends only on the isomorphism type of $V_{j_1,j_2,p}$. It is therefore easy to see that

$$dim(V_{j_1,j_2,p} \cap \Big(\bigcap_{\ell=1}^{k-1} (W_{a^{(\ell)},1} \cap W_{1,b^{(\ell)}})\Big))$$

depends only on the isomorphism type of $V_{j_1,j_2,p}$. By the inductive hypothesis on (2.6), $dim(V_{j_1,j_2,p} \cap f_{(a_{j_1},a_{j_2}),m}) = 1$. Hence, for all $(j_1, j_2) \in J'$ and for all r and p,

$$\begin{aligned} 1 = dim(V_{j_1,j_2,p} \cap f_{(a_{j_1},a_{j_2}),m}) &= dim(V_{j_1,j_2,p} \cap \Big(\bigcap_{\ell=1}^{k-1} (W_{a^{(\ell)},1} \cap W_{1,b^{(\ell)}})\Big)) \\ &= dim(V^{(r)}_{j_1,j_2} \cap \Big(\bigcap_{\ell=1}^{k-1} (W_{a^{(\ell)},1} \cap W_{1,b^{(\ell)}})\Big)) \\ &= dim(V^{(r)}_{j_1,j_2} \cap f_{(a_{j_1},a_{j_2}),m}). \end{aligned}$$

Also, for each specific ordered pair $(\hat{j_1}, \hat{j_2})$ in J', $f_{(a_{\hat{j_1}},a_{\hat{j_2}}),m} \cap V^{(r)}_{j_1,j_2} = 0$ if $(\hat{j_1}, \hat{j_2}) \neq (j_1, j_2)$. Therefore

$$\begin{aligned} f_{(a_{\hat{j_1}},a_{\hat{j_2}}),m} \cap M_r &= \bigoplus_{(j_1,j_2)\in J'} \big(f_{(a_{\hat{j_1}},a_{\hat{j_2}}),m} \cap V^{(r)}_{j_1,j_2}\big) \\ &= f_{(a_{\hat{j_1}},a_{\hat{j_2}}),m} \cap V^{(r)}_{\hat{j_1},\hat{j_2}} \\ &= 1. \end{aligned}$$

Thus, the projections of $f_{(a_{j_1},a_{j_2}),m}$ onto the M_r are disjoint 1-dimensional spaces , and the projections of $f_{(a_{j_1},a_{j_2}),m}$ onto the W_{a_s,b_t} are disjoint P_{a_s,b_t}-dimensional spaces.

A nonzero projection of $f_{(a_{j_1},a_{j_2}),m}$ onto W_{a_s,b_t} is precisely a frequency contained in W_{a_s,b_t}. Thus, (2.3) holds from the argument above. Also, there are $dim(W_{a_s,b_t})/P_{a_s,b_t} = dim(M_r)$ many distinct frequencies in W_{a_s,b_t}, because each distinct nonzero frequency in W_{a_s,b_t} is of dimension P_{a_s,b_t}. Thus, this set of frequencies can be rightfully denoted by $\{f_{(a_s,b_t),w}, 1 \leq w \leq dim(M_r)\}$. (2.2) follows. Also, it follows that

$$\begin{aligned}
\bigoplus_{w=1}^{dim(M_r)} f_{(a_s,b_t),w} &= \bigoplus_{r=1}^{P_{a_s,b_t}} \bigoplus_{w=1}^{dim(M_r)} f_{(a_s,b_t),w} \cap M_r \\
&= \bigoplus_{r=1}^{P_{a_s,b_t}} M_r \\
&= W_{a_s,b_t}.
\end{aligned}$$

Thus, (2.4) holds. The statements from (2.6) through (2.9) follow from a similar argument. □

Recall that the choice of the basis in the example of $\mathbb{C}S_3$ was natural because irreducibles at each step were multiplicity free in all double coset spaces. If there exists an irreducible with multiplicity in a double coset, DCFs projected onto an isotypic (e.g $e_{x\vdash k}DCF$) will not be one dimensional. For example, suppose that a double coset space $\mathbb{C}S_k g S_{k+1}$ contains an irreducible corresponding to an ordered pair of partitions $(a \vdash k, b \vdash (k+1))$ with multiplicity 2. Then by Theorem 2.1, each frequency contained in (a, b) will have a 2-dimensional intersection with the double coset space $\mathbb{C}S_k g S_{k+1}$. (Also recall from Theorem 2.1 that there are as many frequencies in (a, b) as the dimension of an irreducible corresponding to (a, b)). Consider any DCF contained in $S_k g S_{k+1}$, whose corresponding pair of partitions $(a \vdash k, b' \vdash k)$ is such that $b' \vdash k$ has a directed edge to $b \vdash (k+1)$ in the Bratteli diagram. Then the projection of this DCF to a right $\mathbb{C}S_{k+1}$-isotypic space corresponding to b will have exactly dimension 2. So we will have to make a decision regarding the basis that spans this 2-dimensional space. Unfortunately, many double cosets in S_n for larger n (≥ 4) contain irreducible bimodules with multiplicities. This fact was shown first by Eric Malm '05. The team of Brad Froehle and Marie Jameson '07 used the Gram-Schmidt method to determine the basis for the multiple dimensional projection of DCFs. In Chapter 3, we will computate the multiplicity of each irreducible bimodule in each double coset. In Chapter 4, we shall also mention a particular construction of an intermediate basis in recent joint work with Mike Hansen '07.

2.3 Orrison-DIF and Clausen-DIT : Matrix-Form and Sum-Form

This section is dedicated to a comparative study of the ODIF, in which I compare the ODIF to a particular decimation in time (DIT) algorithm developed by Michael Clausen. This comparative study has not only inferred a reason for choosing DIF over DIT for our study, but also reveals a connection between DIT and DIF.

A comparison between the Orrison DIF and Clausen's decimation in time (Clausen-DIT) highlights the importance of distinguishing two ways of presenting FFT algorithms: **sum-form** and **matrix-form**. The ODIF is presented in matrix form, and ClausenDIT is presented in the sum-form. Each way of expression has its own advantages and disadvantages, hiding and disclosing different types of information. In particular, the matrix-form is particularly useful for studying the decomposition of modules, and the sum-form is more suitable for inventing techniques that can speed up the algorithm. Many researchers of the FFT prefer the sum-form in developing their algorithms. However, with this presentation of the algorithm, the underlining decompositions of the algebra are completely concealed. However, if one can convert the sum-form to the matrix-form and vice versa, one will be able to know the algebraic significance of the techniques, allowing for much deeper articulation of the algorithm and understanding of the subject. For example, converting the Clausen-DIT, which is defined below, to matrix-form revealed that the Clausen-DIT is equivalent to the OS-ODIF from module theoretic standpoint. Let us first present the Clausen-DIT.

2.3.1 The Clausen-DIT

Let $\{D_i\}_{i=1}^{h}$ be the set of all irreducible representations of S_n, and suppose a is an element in the algebra $\mathbb{C}S_n$. Let $\{g_j\}_{j=1}^{n}$ be the transversals of the left S_{n-1} cosets in S_n. Then every $a \in \mathbb{C}S_n$ can be written as $a = \sum_{j=1}^{n} g_j a_j$ with $a_j \in \mathbb{C}S_{n-1}$ (each a_j is a formal sum of elements in S_{n-1}). Further, let ϕ_n : $\mathbb{C}S_n \to \bigoplus_{i=1}^{h} \mathbb{C}^{d_i \times d_i}$ be a Wedderburn's isomorphism with respect to the left adapted basis under which the representations of $\mathbb{C}S_k$ for all isomorphic

$\mathbb{C}S_k$-irreducibles are equal for all $k \leq n$.[3] Then

$$\begin{aligned}
\phi_n(a) &= \bigoplus_i D_i(a) \\
&= \bigoplus_i D_i(\sum_{j=1}^{[S_n:S_{n-1}]} g_j a_j) \\
&= \sum_{j=1}^{[S_n:S_{n-1}]} \bigoplus_i D_i(g_j)(D_i \downarrow S_{n-1})(a_j).
\end{aligned}$$

Therefore, we can compute $\phi_n(a)$ recursively as follows:

The Clausen-DIT Algorithm

1. *Apply the DFT of $\mathbb{C}S_n$ to a_j for all j using the Clausen-DIT to obtain $D_i \downarrow S_{n-1}(a_j)$ for all i, j.*
2. *Multiply $D_i \downarrow S_{n-1}(a_j)$ by precomputed $D_i(g_j)$ for each i, j.*

Since the fomulation above is in the form of a sum, we say that the algorithm is in a **sum-form**.

From a module theoretic standpoint, Clausen's algorithm does the following actions in order:

- decimates $\mathbb{C}S_n$ into left-coset spaces $\{\mathbb{C}g_jS_{n-1}; j = 1, \ldots [S_n : S_{n-1}]\}$.
- applies $DFT_{S_{n-1}}$ to each left-coset space $\mathbb{C}gS_{n-1}$ and projects the result to the right-$\mathbb{C}S_{n-1}$-isotypic spaces.
- projects the results from the second step to the right-$\mathbb{C}S_n$-isotypic spaces.

This fact is not so easy to see in the sum-form. However, we will show that this is exactly what is happening, by converting the algorithm into a factorization of the DFT matrix (**matrix-form**).

Lemma 2.1. *Assume that the ordering of the standard basis in $\mathbb{C}S_n$ respects $\mathbb{C}S_n$'s decomposition into left S_i coset spaces for all S_i in the subgroup chain $S_1 \leq S_2 \cdots \leq S_n$. Then Clausen's Algorithm is equivalent to the factorization of the DFT into the matrices defined recursively as*

$$DFT_{S_n} = [A_1 \ldots A_{[S_n:S_{n-1}]}]\big(I_{[S:S_{n-1}]} \otimes DFT_{S_{n-1}}\big),$$

[3] This is the Clausen's adapted basis mentioned in Chapter 1.

where each A_j is a matrix of dimension $|S_n| \times [S_n : S_{n-1}]$, which can be constructed from $\phi_n(g_j) = \bigoplus_i D_i(g_j)$.

Proof. Let $[g_j a_j]_{g_j S_{n-1}}$ be the vector form of $g_j a_j$ over $\mathbb{C}$ in the standard basis of $g_j S_{n-1}$. Note that, as $|S_{n-1}|$-dimensional vectors over $\mathbb{C}$, $[g_j a_j]_{g_j S_{n-1}} = [a_j]_{S_{n-1}}$. Also, the entries of

$$\bigoplus_i D_i(g_j)(D_i \downarrow S_{n-1})(a_j)$$

are precisely that of $\phi_n(g_j) * \phi_n(a_j)$. We can obtain $\phi_n(g_j) * \phi_n(a_j)$ in steps. First, observe that $\phi_n(a_j) = (\bigoplus_i D_i \downarrow S_{n-1})(a_j)$ is a direct sum of irreducible representations of $\mathbb{C}S_{n-1}$. Because the blocks in $\phi_{n-1}(\mathbb{C}S_{n-1})$ contain all irreducible representations of $\mathbb{C}S_{n-1}$, we can obtain $\phi_n(a_j)$ by copying the blocks in $\phi_{n-1}(a_j)$. Denote this 'copying map' from $\phi_{n-1}(\mathbb{C}S_{n-1})$ to $\phi_n(\mathbb{C}S_{n-1})$ by γ. We can then obtain $\phi_n(g_j) * \phi_n(a_j)$ by multiplying $\phi_n(a_j)$ and $\phi_n(g_j)$ in the frequency domain. Define a map $\alpha : \phi_n(\mathbb{C}S_{n-1}) \mapsto \phi_n(\mathbb{C}S_n)$ by $\alpha(\phi_n(a)) = \phi_n(g_j) * \phi_n(a)$. Thus we can compute $\phi_n(g_j) * \phi_n(a_j)$ from $[g_j a_j]_{g_j S_{n-1}}$ in following chain of maps:

$$[g_j a_j]_{g_j S_{n-1}} \xrightarrow{DFT_{S_{n-1}}} \phi_{n-1}(a_j) \xrightarrow{\gamma} \phi_n(a_j) \xrightarrow{\alpha} \phi_n(g_j) * \phi_n(a_j)$$

It is clear that α, γ are both linear transformations. Thus the composition of maps $\alpha \circ \gamma$ is a linear transformation that maps a $|S_{n-1}|$-dimensional vector $DFT_{S_{n-1}}[a_j]$ to a $|S_n|$-dimensional vector $\phi_n(g_j) * \phi_n(a_j)$, and is hence expressible by a matrix A_j of dimension $|S_n| \times [S_n : S_{n-1}]$. It is also clear that the entries of A_j are from $\phi_n(g_j)$, because γ only involves copying of the entries. Thus

$$\left(DFT_{S_n} \downarrow \mathbb{C}g_j S_{n-1}\right)[g_j a_j] = A_j DFT_{S_{n-1}}[a_j]_{S_{n-1}} = A_j DFT_{S_{n-1}}[g_j a_j]_{g_j S_{n-1}},$$

and the claim follows. □

2.3.2 Example

Below, we present the matrix form of Clausen's algorithm for S_3 and S_4. Let $D_{a \vdash k}$ be an irreducible representation of S_k. Also, let $D(x)_{ij}$ be the ijth entry of $D(x)$. Then we can see from the Bratteli diagram that

$$D_{(3)} \downarrow S_2 = D(2), \quad D_{(2,1)} \downarrow S_2 = \begin{pmatrix} D_{(2)} & \\ & D_{(1,1)} \end{pmatrix}, \quad D_{(1,1,1)} \downarrow S_2 = D(1,1)$$

Then

$$DFT_{S_3} = [A(1) \quad A(23) \quad A(13)] \begin{pmatrix} DFT_{S_2} & & \\ & DFT_{S_2} & \\ & & DFT_{S_2} \end{pmatrix},$$

where

$$A(x) = \begin{pmatrix} D_{(3)}(x) & \\ D_{(2,1)}(x)_{11} & \\ D_{(2,1)}(x)_{21} & \\ & D_{(2,1)}(x)_{21} \\ & D_{(2,1)}(x)_{22} \\ & D_{(1,1,1)}(x) \end{pmatrix}.$$

Also,

$$D_{(2,1,1)} \downarrow S_3 = \begin{pmatrix} D_{(1,1,1)} & \\ & D_{(2,1)} \end{pmatrix}$$

and

$$D_{(3,1)} \downarrow S_3 = \begin{pmatrix} D_{(2,1)} & \\ & D_{(3)} \end{pmatrix}.$$

Then

$$DFT_{S_4} = [B(1) \quad B(14) \quad B(24) \quad B(34)] \begin{pmatrix} DFT_{S_3} & & & \\ & DFT_{S_3} & & \\ & & DFT_{S_3} & \\ & & & DFT_{S_3} \end{pmatrix}$$

where $B(x)$ is

$$\begin{pmatrix}
D_{(4)}(x) & & & & & \\
 & & & & & D_{(2,1,1)}(x)_{11} \\
 & & & & & D_{(2,1,1)}(x)_{21} \\
 & & & & & D_{(2,1,1)}(x)_{31} \\
 & D_{(2,1,1)}(x)_{12} & D_{(2,1,1)}(x)_{13} & & & \\
 & D_{(2,1,1)}(x)_{22} & D_{(2,1,1)}(x)_{23} & & & \\
 & D_{(2,1,1)}(x)_{32} & D_{(2,1,1)}(x)_{33} & & & \\
 & & & D_{(2,1,1)}(x)_{12} & D_{(2,1,1)}(x)_{13} & \\
 & & & D_{(2,1,1)}(x)_{22} & D_{(2,1,1)}(x)_{23} & \\
 & & & D_{(2,1,1)}(x)_{32} & D_{(2,1,1)}(x)_{33} & \\
 & D_{(2,2)}(x)_{11} & D_{(2,2)}(x)_{12} & & & \\
 & D_{(2,2)}(x)_{21} & D_{(2,2)}(x)_{22} & & & \\
 & & & D_{(2,2)}(x)_{11} & D_{(2,2)}(x)_{12} & \\
 & & & D_{(2,2)}(x)_{21} & D_{(2,2)}(x)_{22} & \\
 & D_{(3,1)}(x)_{12} & D_{(3,1)}(x)_{13} & & & \\
 & D_{(3,1)}(x)_{22} & D_{(3,1)}(x)_{23} & & & \\
 & D_{(3,1)}(x)_{32} & D_{(3,1)}(x)_{33} & & & \\
 & & & D_{(3,1)}(x)_{12} & D_{(3,1)}(x)_{13} & \\
 & & & D_{(3,1)}(x)_{22} & D_{(3,1)}(x)_{23} & \\
 & & & D_{(3,1)}(x)_{32} & D_{(3,1)}(x)_{33} & \\
D_{(3,1)}(x)_{11} & & & & & \\
D_{(3,1)}(x)_{21} & & & & & \\
D_{(3,1)}(x)_{31} & & & & & \\
 & & & & & D_{(1,1,1,1)}(x)
\end{pmatrix}.$$

We can then make the following critical observation:

Theorem 2.2. *OS-ODIF and Clausen's algorithm decimate $\mathbb{C}S_n$ in the same way, if the basis for $DFT_{S_{n-1}}$ used in the Clausen's algorithm is doubly adapted.*

Proof. Suppose v is a vector in a $(\mathbb{C}S_k, \mathbb{C}S_k)$ doubly adapted basis B of $\mathbb{C}S_n$ that is contained in S_k (a DFT_{S_k} basis). Then v spans the 1-dimensional vector space given by

$$\mathbb{C}S_k \cap \Big(\bigcap_{i=1}^{k} (W_{a^{(i)} \vdash i, (1) \vdash 1} \cap (W_{(1) \vdash 1, b^{(i)} \vdash i}) \Big)$$

where $\{a^{(i)}\}_{i=1}^{k}, \{b^{(i)}\}_{i=1}^{k}$ are unique series of partitions such that $a^{(i)}, b^{(i)} \vdash i$. This implies that for any $i \leq k$, $v = e_{a^{(i)}} v = v e_{b^{(i)}}$. Let $\{g_j\}_{j \in J}$ be the set of transversals of left S_k cosets in S_n. The above implies that $g_j v e_{b^{(i)}} = g_j v$ for all j. Thus, $g_j v$ is in a space

$$g_j \mathbb{C} S_k \cap \Big(\bigcap_{i=1}^{k} (W_{(1)\vdash 1, b^{(i)} \vdash i}) \Big).$$

Also, it is clear that two vectors v_1, v_2 are $\mathbb{C}$-linearly independent if and only if $g_j v_1, g_j v_2$ are $\mathbb{C}$-linearly independent. Thus if B' is a subset of B that spans

$$\mathbb{C} S_k \cap \Big(\bigcap_{i=1}^{k} (W_{(1)\vdash 1, b^{(i)} \vdash i}) \Big),$$

then $g_j B'$ spans

$$g_j \mathbb{C} S_k \cap \Big(\bigcap_{i=1}^{k} (W_{(1)\vdash 1, b^{(i)} \vdash i}) \Big).$$

From the lemma 2.1, $\bigcup_{j \in J} g_j B$ is an intermediate basis of $\mathbb{C} S_n$ at the kth level of the recursion (the basis after applying $I_{[S_n : S_k]} \otimes DFT_{S_k}$). Now, suppose J' is a subset of J such that $\bigcup_{j \in J'} g_j S_k$ is a left S_{k+1} coset. Note that $\bigcup_{j \in J'} g_j B'$ spans a DCF in the OS-ODIF. $\square$

The conversion from sum-form to matrix-form shows that, from a module-theoretic standpoint, the difference between DIT and DIF is subtle. The equivalence between the OS-ODIF and Clausen-DIT is significant. Past results in this research indicate that the regular ODIF is much faster than the OS-ODIF. On the other hand, the currently fastest FFT algorithm by David Maslen (1998) stems from the Clausen-DIT. This suggests the possibility that the ODIF may be modified to become faster than Maslen's algorithm.

Chapter 3

Analysis of Permutation Bimodules

3.1 DCFs and Double Coset Spaces

As stated in Chapter 2, each step of the ODIF is a collection of the change of bases in each DCF. To this end, the factor in the factorization of the DFT that corresponds to a particular step in the ODIF will be a block diagonal matrix $\bigoplus_{\text{DCF in the step}} End_{\mathbb{C}}(DCF)$. Therefore knowing the size and the number of DCFs at each step allows us to predict the exact number of blocks that will appear in each factorization and their sizes. We use this information to provide a bound to the runtime of the ODIF algorithm.

Recall that DCF is an intersection of a frequency and a double coset space. Hence I may write any DCF at the 2nd task of the kth step in the ODIF as $\mathbb{C}S_k g S_{k+1} \cap f_{(a\vdash k, b\vdash k),m}$. In particular, if $\{S_k g_i S_k\}_{i\in I}$ is a collection of double cosets such that

$$\bigcup_{i\in I} S_k g_i S_k = S_k g S_{k+1}.$$

I can also write any DCF in $S_k g S_{k+1}$ as

$$\bigcup_{i\in I} \left(\mathbb{C}S_k g_i S_k \cap f_{(a\vdash k, b\vdash k),m}\right).$$

We will compute the dimension of this DCF. As in the previous chapter, let $W_{a\vdash k, b\vdash k}$ denote the $(\mathbb{C}S_k, \mathbb{C}S_k)$ isotypic space corresponding to the pair of partitions (a, b). By Theorem 2.1, $f_{(a,b),m}$ has a 1-dimensional intersection with any ($\mathbb{C}S_k, \mathbb{C}S_k$) bimodule irreducible in the isotypic space $W_{a,b}$.

Therefore the dimension of $\mathbb{C}S_k g_i S_k \cap f_{(a\vdash k,b\vdash k),m}$ is precisely **the multiplicity of irreducible ($\mathbb{C}S_k, \mathbb{C}S_k$)-bimodules in $\mathbb{C}S_k g_i S_k$ that correspond to the ordered pair of partitions (a, b).** Thus we need to know the following information in order to know the size of each DCF.

- A precise decomposition of each (S_k, S_{k+1}) double coset into (S_k, S_k) double cosets.
- The multiplicities of irreducible ($\mathbb{C}S_k, \mathbb{C}S_k$)-bimodules in each (S_k, S_k) double coset space

3.2 Classification of Double Coset Spaces

Throughout this section, let k and h be integers such that $k \leq h \leq k+1$. In this section, I will classify all the (S_k, S_h) double coset spaces by their isomorphism types as ($\mathbb{C}S_k, \mathbb{C}S_h$)- bimodules. This classification plays a pivotal role in obtaining the desired information mentioned above. I will begin with the following useful fact.

Theorem 3.1. *Any pair of (S_k, S_h) double coset spaces in S_n of the same dimension are isomorphic as $(\mathbb{C}S_k, \mathbb{C}S_h)$ bimodule.*

Proof. Given a double coset $S_k g S_h$, $gS_h g^{-1} \cap S_k$ is a subgroup of S_k isormophic to S_ℓ for some $1 \leq \ell \leq k$. I claim that there exists a double coset representative $\bar{g}$ in $S_k g S_h$ such that $\bar{g} S_h \bar{g}^{-1} \cap S_k = S_\ell$ and $\bar{g}$ commutes with elements in S_ℓ. We call $\bar{g}$ a canonical double coset representative. Let $\bullet$ denote the natural group action of the symmetric group (e.g. $(123) \bullet 2 = 3$). Then note that $gS_h g^{-1}$ is a symmetric group with support

$$g \bullet 1, g \bullet 2, \ldots, g \bullet h$$

which shares ℓ elements with the set $\{1, 2, \ldots k\}$. Now, for any element $\kappa \in S_k$,

$$\left|\{\kappa g \bullet 1, \kappa g \bullet 2, \ldots, \kappa g \bullet h\} \cap \{1, 2, \ldots k\}\right| = \ell.$$

For any set A of size ℓ in $\{1, 2, \ldots k\}$, S_k contains an element that can map A to $\{1, 2, \ldots \ell\}$. In particular, there exists an element $\delta \in S_k$ such that

$$\{\delta g \bullet 1, \delta g \bullet 2, \ldots, \delta g \bullet h\} \cap \{1, 2, \ldots k\} = \{1, 2, \ldots \ell\}.$$

Thus there exists a double coset representative $g' = \delta g$ such that $g' S_h g'^{-1} \cap S_k = S_\ell$. Also note that, for any element $\eta \in S_h$,

$$\{g'\eta \bullet 1, g'\eta \bullet 2, \ldots, g'\eta \bullet h\} = \{g' \bullet 1, g' \bullet 2, \ldots, g' \bullet h\}.$$

Because $\{1, 2, \ldots \ell\} \subset \{g' \bullet 1, g' \bullet 2, \ldots, g' \bullet h\}$, it is evident that there exists an element $\sigma \in S_h$ such that $g'\sigma \bullet i = i$ for all $i \in \{1, 2, \ldots \ell\}$. That is, $g'\sigma$ commutes with elements in S_ℓ. Let $g'\sigma = \bar{g}$. $\bar{g}$ is a canonical double coset representative.

Suppose $|S_k g_1 S_h| = |S_k g_2 S_h|$. Without loss of generality, assume that g_1 and g_2 are both canonical double coset representative. Hence $g_1 S_h g_1^{-1} \cap S_k = g_2 S_h g_2^{-1} \cap S_k = S_\ell$, and g_1 and g_2 both commute with elements in S_ℓ. Consider the map

$$\phi : S_k g_1 S_h \to S_k g_2 S_h \quad \text{defined by} \quad \phi(\kappa g_1 \eta) = \kappa g_2 \eta.$$

I will show that this map is well defined.

If $\kappa g_1 \eta = \hat{\kappa} g_1 \hat{\eta}$, then $\hat{\kappa}^{-1}\kappa = g_1 \hat{\eta} \eta^{-1} g_1^{-1}$. Clearly, $\hat{\kappa}^{-1}\kappa \in g_1 S_h g_1^{-1} \cap S_k = S_\ell$. Now, see that

$$g_2 g_1^{-1}(\hat{\kappa}^{-1}\kappa) g_1 g_2^{-1} \quad = \quad g_2(g_1^{-1} g_1 \hat{\eta} \eta^{-1} g_1^{-1}) g_1 g_2^{-1}.$$

because $\hat{\kappa}^{-1}\kappa \in g_1 S_h g_1^{-1} \cap S_k$, $\hat{\kappa}^{-1}\kappa$ commutes with both g_1 and g_2. Thus

$$\hat{\kappa}^{-1}\kappa = g_2 \hat{\eta} \eta^{-1} g_2^{-1}.$$

Organizing, we get

$$\kappa g_2 \eta = \hat{\kappa} g_2 \hat{\eta}.$$

Injectivity follows from similar argument, and surjectivity is trivial. It is evident that ϕ can be extended to the isomorphism between $\mathbb{C}S_k g_1 S_h$ and $\mathbb{C}S_k g_2 S_h$. □

Therefore, the bimodule isomophism type of a double coset space $\mathbb{C}S_k g S_h$ is determined solely by ℓ such that $|gS_h g^{-1} \cap S_k| = |S_\ell|$. We may call ℓ the **type** of the double coset space. With this theorem, we can simplify the problem of classifying the bimodule-isomorphism-types of all (S_k, S_h) double coset spaces into classifying the (S_k, S_h) double cosets by their sizes. The following theorem determines all possible sizes of (S_k, S_h) double cosets in S_n. The theorem was proven independently by the author and Brad Froehle of the University of Minnesota. For notational reasons, I will present here Froehle's version of the proof.

Theorem 3.2. *For every ℓ such that $max(0, k+h-n) \leq \ell \leq k$, there exists at least one double coset of size $k!h!/\ell!$. Moreover, every (S_k, S_h) double coset in S_n is of size $k!h!/\ell!$ for some ℓ satisfying $max(0, k+h-n) \leq \ell \leq k$.*

Proof. Given a double coset $S_k g S_h$, its size is the same as that of $S_k g S_h g^{-1}$, which is a product of two subgroups of S_n, namely S_k and $gS_h g^{-1}$. Therefore its size is given by $|S_k||gS_h g^{-1}|/|S_k \cap gS_h g^{-1}| = |S_k||S_h|/|S_k \cap gS_h g^{-1}|$. As mentioned in the proof of Theorem 3.1, $S_k \cap gS_h g^{-1}$ is a symmetric subgroup of S_k isomorphic to S_ℓ for some ℓ, with support

$$\{g \bullet 1, g \bullet 2, \ldots g \bullet h\} \cap \{1, 2, \ldots k\}.$$

First, it is obvious that the size of the set above is bounded by k. Now, because $g \bullet i$ is a distinct element in $\{1,2, \ldots \text{n}\}$ for all $i \in \{1, 2, \ldots h\}$, the set above cannot be smaller than $max(0, k+h-n)$. Finally, if ℓ is any number satisfying $max(0, k+h-n) \leq \ell \leq k$, let g be a cycle

$$(\ell+1, h+1)(\ell+2, h+2) \cdots (k, h+(k-\ell)).$$

Then $\{g \bullet 1, g \bullet 2, \ldots g \bullet H\} \cap 1, 2, \ldots k = \{1, 2, \ldots \ell\}$. Note that $0! = 1!$. Therefore the theorem states that the range of *types* of S_k, S_h double cosets in S_n is $max(1, k+h-n) \leq \ell \leq k$. □

Fortunately, Brad Froehle and Marie Jameson in their research during Summer 2006 found a formula to count the number of (S_k, S_h) double cosets of certain sizes in S_n.

Theorem 3.3. (Froehle, Jameson) *Denote the number of (S_k, S_h) double cosets of size $k!h!/\ell!$ in S_n by $m(S_k, S_h, \ell)_{S_n}$. Also, let*

$$f(k, h, \ell, n) = \frac{(n-h)!(n-k)!}{(k-\ell)!(h-\ell)!(n-h-k+\ell)!}.$$

Then

$$m(S_k, S_h, \ell)_{S_n} = \begin{cases} f(k, h, \ell, n) & \text{if } \ell \neq 1 \\ f(k, h, \ell, 0) + f(k, h, \ell, 1) & \text{if } \ell = 1 \end{cases}$$

3.2.1 Example

Consider the double cosets of S_4. Theorem 3.2 and 3.2 can be verified.

(S_1, S_2) **double cosets**

- Range of the types: $\ell = 1$.
- Double cosets:
  $\{(1), (12)\}, \{(23), (132)\}, \{(13), (123)\}, \{(34), (34)(12)\},$
  $\{(234), (1342)\}, \{(134), (1234)\}, \{(243), (1432)\}, \{(143), (1243)\},$
  $\{(24), (142)\}, \{(14), (124)\}, \{(13)(24), (1423)\}, \{(14)(23), (1324)\}$

(S_2, S_2) **double cosets**

- Range of types : $1 \leq \ell \leq 2$.
- Double cosets
  Type 2 : $\{(1), (12)\}, \{(34), (34)(12)\}$
  Type 1 : $\{(23), (132), (13), (123)\}, \{(234), (1342), (134), (1234)\},$
  $\{(243), (1432), (143), (1243)\}, \{(24), (142), (14), (124)\},$
  $\{(13)(24), (1423), (14)(23), (1324)\}$

(S_2, S_3) **double cosets**

- Range of types : $1 \leq \ell \leq 2$.
- Double cosets
  Type 1 : $\{(1), (12), (23), (132), (13), (123)\},$
  $\{(34), (12)(34), (243), (1243), (1432), (143)\}$
  Type 2: $\{(234), (1234), (1342), (134), (24),$
  $(124), (13)(24), (1324), (142), (14), (1423), (14)(23)\}$

(S_3, S_3) **double cosets**

- Range of types : $2 \leq \ell \leq 3$.
- Double cosets
  Type 3 : $\{(1), (12), (23), (132), (13), (123)\}$
  Type 2 :$\{(34), (12)(34), (243), (1243), (1432), (143)(234), (1234), (1342),$
  $(134), (24), (124), (13)(24), (1324), (142), (14), (1423), (14)(23)\}$

Theorems 3.1, 3.2, and 3.3 together completely classify the (S_k, S_h) double coset spaces by their bimodule isomorphism type. Armed with this classification, let us begin answering the two questions set forth in the first section of this chapter, namely the question regarding the decomposition of double cosets into smaller double cosets.

3.3 Decomposition of Double Cosets

This section will provide the precise decomposition of each (S_k, S_{k+1}) double coset into (S_k, S_k) double cosets. I will begin with a corollary of Theorem 3.1, which states that the type of a double coset also determines how it decomposes into smaller double cosets.

Collorary. *S_kgS_h's decomposition into (S_k, S_{h-1}) (or (S_{k-1}, S_h)) double cosets is determined uniquely by its type. That is, if S_kgS_h and $S_k\bar{g}S_h$ are double cosets of the same type, and if S_kgS_h's decomposition into S_kgS_{h-1} contains m type-ℓ (S_k, S_{h-1}) double cosets for some ℓ, then so does the decomposition of $S_k\bar{g}S_h$.*

Proof. Every (S_k, S_{h-1}) double cosets in S_kgS_h takes the form S_kghS_{h-1} for some $h \in S_h$. The map in the previous theorem from S_kgS_h to $S_k\bar{g}S_h$ maps S_kghS_{h-1} to $S_k\bar{g}hS_{h-1}$. Because the map is bijective, every distinct double coset is mapped to a distinct double coset of same size. The claim follows. □

Finally, let us begin discussing the specifics of the decomposition of the double cosets of a given type.

Theorem 3.4. *The decomposition of a type ℓ double coset contains only type ℓ double cosets and type $\ell - 1$ double cosets.*

Proof. Consider the decomposition of an (S_k, S_k) double coset of type ℓ into (S_{k-1}, S_k) double cosets. Note

$$S_kgS_k = \bigcup_{i}^{k-1} S_{k-1}(ik)gS_k,$$

for transpositions (ik) in S_k. Recall that the cardinality of $S_{k-1} \cap (ik)gS_kg^{-1}(ik)$ determines the type of the double coset space $S_{k-1}(ik)gS_k$. Therefore let us consider $(ik)gS_kg^{-1}(ik)$. Without loss of generality, we can choose g to be a canonical representative of the double coset S_kgS_k. The support of the group gS_kg^{-1} is therefore

$$\{1, \ldots, \ell, w_1, \ldots w_{k-\ell}\},$$

where $w_j \notin \{1, \ldots k\}$ for all $j = 1, \ldots k - \ell$. Then I obtain the support of $(ik)gS_kg^{-1}(ik)$ by replacing i in the set above by k. Hence if $i \in 1, \ldots \ell$, $(ik)gS_kg^{-1}(ik) \cap S_{k-1}$ will have $\ell - 1$ supports, and otherwise, it will have ℓ supports.

Next, consider the decomposition of (S_k, S_{k+1}) double coset into (S_k, S_k) double cosets. Note

$$S_kgS_{k+1} = \bigcup_i S_kg(i(k+1))S_k.$$

Let us consider $g(i(k+1))S_k(i(k+1))g^{-1}$. The support of $gS_{k+1}g^{-1}$ is

$$\{1,\ldots,\ell,w_1,\ldots,w_{k+1-\ell}\},$$

where $w_j \notin \{1,\ldots k\}$ for all $j=1,\ldots k+1-\ell$.

By Theorem 3.1, if we let g to be canonical, $g \bullet r = r$ for all $r \in \{1,\ldots \ell\}$. Then we can assume that $g \bullet (k+1) > k$. To see why, assume otherwise, so $g \bullet (k+1) \in \{1,\ldots k\}$. But only $1,\ldots,\ell$ are the elements of $\{1,\ldots k\}$ in $\{g \bullet 1,\ldots g \bullet (k+1)\} = \{1,\ldots,\ell,w_1,\ldots,w_{k+1-\ell}\}$. Thus $g \bullet (k+1) \in \{1,\ldots k\}$ means that $g \bullet (k+1) \in \{1,\ldots \ell\}$. This is a contradiction.

Thus the support of $g(i(k+1))S_k(i(k+1))g^{-1}$ is given by

$$\{1,\ldots,\ell,w_1,\ldots,w_{k+1-\ell}\}\backslash(g \bullet i).$$

So if $g \bullet i \in \{1,\ldots \ell\}$, $g(i(k+1))S_k(i(k+1))g^{-1} \cap S_k$ has support $\{1,\ldots \ell\}\backslash(g \bullet i) = \{1,\ldots \ell\}\backslash i$. Otherwise, it will have ℓ support. □

Next, let $[S_k,S_h,\ell] : [S_{k'},S_{h'},m]$ denote the number of $(S_{k'},S_{h'})$ double cosets of type m in the decomposition of an (S_k,S_h) double coset of type ℓ into $(S_{k'},S_{h'})$ double cosets. The following theorem is a direct consequence of the previous theorem.

Theorem 3.5.

$$[S_k,S_k,\ell] : [S_{k-1},S_k,m] = \begin{cases} k & \textit{if } \ell = m = 1 \\ k-\ell & \textit{if } \ell = m \neq 1 \\ 1 & \textit{if } m = \ell - 1 \\ 0 & \textit{otherwise} \end{cases}$$

$$[S_k,S_{k+1},\ell] : [S_k,S_k,m] = \begin{cases} k+1 & \textit{if } \ell = m = 1 \\ k-\ell+1 & \textit{if } \ell = m \neq 1 \\ 1 & \textit{if } m = \ell - 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Consider the decomposition of an (S_k,S_k) double coset into S_{k-1},S_k double cosets. By Theorem 3.2 and Theorem 3.4, it is guaranteed that the (S_k,S_k) double coset ot type k decomposes only to (S_k,S_{k-1}) double cosets of type $k-1$. Thus,

$$[S_k,S_k,k] : [S_{k-1},S_k,k] = \frac{k!k!}{k!} / \frac{k!(k-1)!}{(k-1)!} = 1$$

On the other hand, $[S_k,S_k,k-1] : [S_{k-1},S_k,k-1]$ can be found by removing the contribution of the (S_k,S_k) double cosets of type k to the calculation of $m(S_{k-1},S_k,k-1)_{S_n}$ and dividing the result by $m(S_k,S_k,k-1)_{S_n}$, which the number of (S_k, S_k) double coset of type $k-1$ (Theorem 3.3). Simplification reveals that $[S_k, S_k, k-1] : [S_{k-1}, S_k, k-1]$ is given by

$$\frac{m(S_{k-1}, S_k, k-1)_{S_n} - \big([S_k, S_k, k] : [S_{k-1}, S_k, k]\big) m(S_k, S_k, k)_{S_n}}{m(S_k, S_k, k-1)_{S_n}} = 1.$$

Because each double coset decomposes into only two types of double cosets, if we know $[S_k, S_k, \ell] : [S_{k-1}, S_k, \ell]$, we can then compute $[S_k, S_k, \ell] : [S_{k-1}, S_k, \ell - 1]$ by removing the contribution of the (S_{k-1}, S_k) double coset of type ℓ in the ambient (S_k, S_k) double coset and dividing the result by the size of the (S_{k-1}, S_k) double coset of type $\ell - 1$:

$$[S_k, S_k, \ell] : [S_{k-1}, S_k, \ell - 1] = \frac{k!k!/\ell! - (k-1)!k!/\ell!\big([S_k, S_k, \ell] : [S_{k-1}, S_k, \ell]\big)}{(k-1)!k!/(\ell-1)!}.$$

By way of induction, assume that $[S_k, S_k, \ell] : [S_{k-1}, S_k, \ell] = k - \ell$. (Note that $[S_k, S_k, k-1] : [S_{k-1}, S_k, k-1]$ serves as the base case). Substituting this value into (3.1), we get $[S_k, S_k, \ell] : [S_{k-1}, S_k, \ell - 1] = 1$. Likewise, $[S_k, S_k, \ell - 1] : [S_{k-1}, S_k, \ell - 1]$ is given by

$$\frac{m(S_{k-1}, S_k, \ell - 1)_{S_n} - \big([S_k, S_k, \ell] : [S_{k-1}, S_k, \ell - 1]\big) m(S_k, S_k, \ell)_{S_n}}{m(S_k, S_k, \ell - 1)_{S_n}}. \tag{3.1}$$

Substituting $[S_k, S_k, \ell - 1] : [S_{k-1}, S_k, \ell - 1] = 1$, (3.2) becomes

$$\frac{m(S_{k-1}, S_k, \ell - 1)_{S_n} - m(S_k, S_k, \ell)_{S_n}}{m(S_k, S_k, \ell - 1)_{S_n}}. \tag{3.2}$$

Simplification reveals that (3.3) is $k - \ell + 1$ when $\ell \neq 2$, and k when $\ell = 2$. The claim follows. The same sequence of computation also verifies the formula for $[S_k, S_{k+1}, \ell] : [S_k, S_k, m]$. $\square$

3.4 Representation Theory of the Double Coset Space $\mathbb{C}S_k g S_h$

In this section, I answer the second question in Section 3.1; the question regarding the multiplicities of irreducible bimodules in double coset spaces. Suppose that $A_1, A_2, \ldots$ are the irreducible representations of $\mathbb{C}S_h$ and $B_1, B_2, \ldots$ are the irreducible modules of $\mathbb{C}S_k$. Recall that any irreducible $(\mathbb{C}S_k, \mathbb{C}S_h)$ bimodule is isomorphic to $B_i \otimes_{\mathbb{C}} A_j$ for some i, j.

Malm-Matrix

$M_{S_k,S_h,\ell}$ is a matrix for which the ijth entry is the multiplicity of $B_i \otimes_{\mathbb{C}} A_j$ in an (S_k, S_h) double coset space of type ℓ. We organize the multiplicities of irreducible bimodules in double coset spaces with Malm matrices.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Figure 3.1: Malm matrix $M^{S_4}_{S_3,S_3,2}$.

I will present a method for finding all the Malm matrices $M_{S_k,S_h,\ell}$ where $k = h$ or $k = h + 1$. The following three lemmas and the previous remarks about double cosets allows me to compute the Malm matrix recursively.

Lemma 3.1. *Suppose I is a minimal two-sided ideal in $\mathbb{C}G$. Then $I \cong L_i \otimes_{\mathbb{C}} R_j$ for some minimal left sided ideal L_i and some minimal right sided ideal R_j in I.*

Proof. Let

$$D : I \to \mathbb{C}^{n\times n}$$

be the isomorphism as specified by Wedderburn's Theorem. Denote $D^{-1}(E_{ij}) = v_{ij}$. By Theorem 2.7 in Clausen and Baum (1993),

$$R_i = span(v_{i1}, v_{i2}, \dots v_{in}) \quad \text{and} \quad L_j = span(v_{1j}, v_{2j}, \dots v_{nj})$$

are a minimal left sided ideal and a minimal right sided ideal in I, respectively for all i and j. Define then a map

$$\phi : I \to L_j \otimes_{\mathbb{C}} R_i$$

by $\phi(v_{\ell k}) = v_{\ell j} \otimes_{\mathbb{C}} v_{ik}$. The bijection is clear. I will show that this is a bimodule isomorphism. Let $g, h \in \mathbb{C}G$, and

$$g \bullet v_{xy} = \sum_{a=1}^{n} c_a v_{ay} \quad \text{and} \quad v_{xy} \bullet h = \sum_{b=1}^{n} d_b v_{xb}$$

for all choice of x and y. Notice that we are able to assume this because

$L_a \cong L_b$ and $R_s \cong R_t$ for any choice of a, b, s, t. Now note that

$$\begin{aligned} g \bullet \phi(v_{\ell k}) \bullet h &= g \bullet v_{\ell j} \otimes_{\mathbb{C}} v_{ik} \bullet h \\ &= \big(\sum_{a=1}^{n} c_a v_{aj}\big) \otimes_{\mathbb{C}} \big(\sum_{b=1}^{n} d_b v_{ib}\big) \\ &= \sum_{a,b} c_a d_b (v_{aj} \otimes_{\mathbb{C}} v_{ib}) \\ &= \phi\big(\sum_{a,b} c_a d_b v_{ab}\big) \\ &= \phi(g \bullet v_{\ell k} \bullet h). \end{aligned}$$

This proves the claim. □

Next, define the **Total Malm matrix** $M^{S_n}_{S_k,S_h,tot}$ to be the matrix for which the ijth entry is the multiplicity of $B_i \otimes_{\mathbb{C}} A_j$ in S_n. In other words,

$$M^{S_n}_{S_k,S_h,tot} = \sum_{max\{1,k+h-n\} \leq \ell \leq k} m(S_k, S_h, \ell)_{S_n} M_{S_k,S_h,\ell}.$$

It is not difficult to compute $M^{S_n}_{S_k,S_h,tot}$.

Lemma 3.2. *Suppose $D_1, D_2, \ldots$ are the irreducible representation of S_n, $A_1, A_2, \ldots$ are the irreducible representation of S_h, $B_1, B_2, \ldots$ are the irreducible representation of S_k. Then*

$$[M^{S_n}_{S_k,S_h,tot}]_{ij} = \sum_m [D_m : B_i] * [D_m : A_j].$$

Proof. Let e_{B_i} and e_{A_j} be centrally primitive idempotents corresponding to B_i and A_j respectively. Then the multiplicity of $B_i \otimes_{\mathbb{C}} A_j$ in S_n is

$$\frac{dim(e_{B_i} S_n e_{A_j})}{dim(B_i) dim(A_j)}.$$

It is visibly clear in the frequency domain that

$$dim(e_{B_i} S_n e_{A_j}) = \sum_m \big(dim(B_i)[D_m : B_i]\big) * \big(dim(A_j)[D_m : A_j]\big).$$

□

Lemma 3.3. $[M_{S_{k-1},S_k,k-1}]_{ij} = [A_j : B_i]$ *and* $M_{S_k,S_k,k} = I$.

Proof. Note that $S_{k-1}(1)S_k = S_k$ is a double coset of type $k-1$ (see that $(1)S_k(1) \cap S_{k-1} = S_{k-1}$). From Lemma 3.1, we can write $\mathbb{C}S_k$ as

$$\bigoplus_i I_i \cong \bigoplus_i L_i \otimes_{\mathbb{C}} R_i$$

where I_i is a double-sided ideal of S_k. The left-sided ideal L_i, when we restrict the action from left side to S_{k-1}, decomposes to direct sum of irreducible representations of S_{k-1}. In particular, if $L_i' \cong B_i'$ and $L_j \cong A_j$, then

$$\begin{aligned} \bigoplus_i I_i &\cong \bigoplus_{i,j}([L_j : L_i']L_i') \otimes_{\mathbb{C}} R_j \\ &\cong \bigoplus_{i,j}[A_j : B_i](L_i' \otimes_{\mathbb{C}} R_j) \end{aligned}$$

The first claim follows. As for $M_{S_k,S_k,k}$, the claim is obvious because

$$A_i \otimes_{\mathbb{C}} A_j = \begin{cases} A_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

□

Computation of Malm Matrices

Theorem 3.1 implies that, in S_n, there are only two types of (S_{n-1}, S_{n-1}) double cosets and only two types of (S_{n-2}, S_{n-1}) double cosets: type $n-1$ double cosets and type $n-2$ double cosets for the former and type $n-2$ double cosets and type $n-3$ double cosets for the latter. Therefore we can say

$$\begin{aligned} M_{S_{n-1},S_{n-1},n-2} &= \frac{M^{S_n}_{S_{n-1},S_{n-1},tot} - m(S_{n-1}, S_{n-1}, n-1)_{S_n} M_{S_{n-1},S_{n-1},n-1}}{m(S_{n-1}, S_{n-1}, n-2)} \\ M_{S_{n-2},S_{n-1},n-3} &= \frac{M^{S_n}_{S_{n-2},S_{n-1},tot} - m(S_{n-2}, S_{n-1}, n-2)_{S_n} M_{S_{n-2},S_{n-1},n-2}}{m(S_{n-2}, S_{n-1}, n-3)}. \end{aligned}$$

We can thus compute $M_{S_k,S_h,k}$ and $M_{S_k,S_h,k-1}$ where $k = h$ or $k+1 = h$.

The range of the types of (S_k, S_k) double cosets that exists in S_n is $max\{1, h + k - n\} \leq \ell \leq k$. Using the $M_{S_k,S_h,k}$ and $M_{S_k,S_h,k-1}$ as base case, we can take advantage of this information to recursively compute Malm matrices for all types of (S_k, S_h) double cosets when $k \leq h \leq k+1$. In particular, we exploit the fact that the range of the types of (S_k, S_h) double cosets in S_{n+1} is at most one greater than the range of the types of (S_k, S_h) double cosets in S_n. To see this, suppose we want to compute $M_{S_k,S_h,\ell}$, and that we already know $M_{S_k,S_h,m}$ for all $\ell + 1 \leq m \leq k$. Note that $S_{h+k-\ell}$ is a symmetric group such that the range of the types of the (S_k, S_h) double cosets is $\ell, \ldots k$. Therefore

$$M_{S_k,S_h,\ell} = \frac{1}{m(S_k, S_h, \ell)_{S_{h+k-\ell}}} \Big(M^{S_{h+k-\ell}}_{S_k,S_h,tot} - \sum_{i=\ell+1}^{k} m(S_k, S_h, i)_{S_{h+k-\ell}} M_{S_k,S_h,i} \Big).$$

Now we know $M_{S_k,S_h,m}$ for all $\ell \leq m \leq k$. Note that $S_{h+k-\ell+1}$ is a symmetric group such that the range of the types of the (S_k, S_h) double cosets of type is $\ell - 1, \ldots k$. We can thus repeat the process to find $M_{S_k,S_h,\ell-1}$.

3.5 The Size of Each DCF

Finally, we can compute the size of each DCF. Again, each DCF is an intersection of a double coset space and a frequency. As in the first section of this chapter, let $\{S_k g_i S_k\}_{i \in I}$ be a collection of double cosets such that

$$\bigcup_{i \in I} S_k g_i S_k = S_k g S_{k+1}.$$

Then

$$\bigcup_{i \in I} \left(\mathbb{C} S_k g_i S_k \cap f_{(a \vdash k, b \vdash k), m} \right)$$

is a DCF. Recall that the dimension of $\left(\mathbb{C} S_k g_i S_k \cap f_{(a \vdash k, b \vdash k), m} \right)$ is the multiplicity of the irreducible $(\mathbb{C}S_k, \mathbb{C}S_k)$-bimodule in $\mathbb{C}S_k g_i S_k$ corresponding to the ordered pair of partitions $(a \vdash k, b \vdash k)$.

Suppose that $S_k g S_{k+1}$ is a double coset of type ℓ. We know that $\{S_k g_i S_k\}_{i \in I}$ consists of $[S_k, S_{k+1}, \ell] : [S_k, S_k, \ell]$ many (S_k, S_k) double cosets of type ℓ and $[S_k, S_{k+1}, \ell] : [S_k, S_k, \ell - 1]$ many (S_k, S_k) double cosets of type $\ell - 1$. The multiplicity of irreducible $(\mathbb{C}S_k, \mathbb{C}S_k)$-bimodules corresponding to the pair of partitions $(a \vdash k, b \vdash k)$ in (S_k, S_k) double coset spaces of type ℓ is given by the entry of the Malm matrix $M_{S_k,S_k,\ell}$ corresponding to $(a \vdash k, b \vdash k)$.

Suppose that the ijth entry of the matrix corresponds to $(a \vdash k, b \vdash k)$. Then, likewise, the multiplicity of $(\mathbb{C}S_k, \mathbb{C}S_k)$ irreducible bimodules corresponding to the pair of partitions $(a \vdash k, b \vdash k)$ in (S_k, S_k) double coset spaces of type $\ell - 1$ is given by the ijth entry of $M_{S_k,S_k,\ell-1}$. Therefore, the dimension of $\bigcup_{i \in I} \left(\mathbb{C} S_k g_i S_k \cap f(a \vdash k, b \vdash k), m \right)$ is given by

$$[S_k, S_{k+1}, \ell] : [S_k, S_k, \ell][M_{S_k,S_k,\ell}]_{ij} + [S_k, S_{k+1}, \ell] : [S_k, S_k, \ell - 1][M_{S_k,S_k,\ell}]_{ij}. \quad (3.3)$$

In the next chapter, I will derive the formula for a crude operation count of the ODIF using (3.4).

Chapter 4

Crude Runtime Bound and an Introduction to the Tensor Basis

We are now ready to compute a crude bound for the operation count of the ODIF for $\mathbb{C}S_n$. I will compare this crude bound against the experimental data generated by others who studied the ODIF, and discuss an interesting problem that arises.

4.1 A Crude Bound of the ODIF

Recall that each step of the ODIF is a collection of change of bases inside the DCFs. The change of basis matrix associated with a given DCF will indeed have size $dim(DCF) \times dim(DCF)$. Multiplying the $dim(DCF)$-dimensional vector by the matrix of this dimension can be implemented in $2dim(DCF)^2$ operations. Ignoring the constant 2, the operation count of the ODIF for S_n is given by

$$\underbrace{\sum_{i=1}^{n}}_{\text{Steps of the ODIF}} \underbrace{\sum_{j=1,2}}_{\text{The two tasks in each step}} \sum_{\text{DCF at } j\text{th task of } i\text{th step}} dim(DCF)^2. \quad (4.1)$$

Let us consider the DCFs at the first task of the kth step. Each DCF in the double coset space $\mathbb{C}S_k g S_{k+1}$ is an intersection of $\mathbb{C}S_k g S_{k+1}$ and a frequency $f_{(a\vdash k, b\vdash k),m}$ for some a, b. Recall from Chapter 2 that an irreducible $(\mathbb{C}S_k, \mathbb{C}S_k)$ bimodule corresponding to $(a \vdash k, b \vdash k)$ is isomorphic to $a \vdash k \otimes_{\mathbb{C}} b \vdash k$. Thus the dimension of a $(\mathbb{C}S_k, \mathbb{C}S_k)$ bimodule corresponding to $(a \vdash k, b \vdash k)$ is given by $dim(a \vdash k)dim(b \vdash k)$.

We know from Theorem 2.1 that the range of m for $f_{(a\vdash k, b\vdash k),m}$ is

$$\{1, \ldots, dim(a \vdash k) dim(b \vdash k)\}.$$

Hence, the number of DCFs contained in the intersection of any given (S_k, S_{k+1}) double coset space and the $(\mathbb{C}S_k, \mathbb{C}S_k)$-isotypic space corresponding to $(a \vdash k, b \vdash k)$ is $dim(a \vdash k)dim(b \vdash k)$.

We also know from (3.4) that, if the double coset $S_k g S_{k+1}$ is of type ℓ, and if the ijth entry of the Malm matrix for (S_k, S_k) double coset space corresponds to $(a \vdash k, b \vdash k)$, the dimension of each DCF in the intersection of $\mathbb{C}S_k g S_{k+1}$ and the $(\mathbb{C}S_k, \mathbb{C}S_k)$-isotypic space corresponding to $(a \vdash k, b \vdash k)$ is

$$[S_k, S_{k+1}, \ell] : [S_k, S_k, \ell][M_{S_k,S_k,\ell}]_{ij} + [S_k, S_{k+1}, \ell] : [S_k, S_k, \ell - 1][M_{S_k,S_k,\ell}]_{ij}.$$

Denote the value above by $R_{(a\vdash k, b\vdash k),\ell}$. Then

$$\begin{aligned}
\sum_{\text{DCF at 1st task of } k\text{th step}} dim(DCF)^2 &= \\
&= \sum_{max\{0,k+(k+1)-n\} \le \ell \le k} \left(m(S_k, S_{k+1}, \ell) \sum_{\text{DCFs in type } \ell \text{ double coset space}} dim(DCF)^2 \right) \\
&= \sum_{max\{0,k+(k+1)-n\} \le \ell \le k} \left(m(S_k, S_{k+1}, \ell) \sum_{a\vdash k, b\vdash k} dim(a \vdash k) dim(b \vdash k) R^2_{(a\vdash k, b\vdash k),\ell} \right).
\end{aligned}$$

This formula correctly predicts the number of blocks and their sizes in the factorization of the DFT matrix in Malm (2005).

The operation count given in (4.1) is a crude bound because the operation count of multiplying a $dim(DCF)$-dimensional vector by a matrix of dimension $dim(DCF) \times dim(DCF)$ is bounded from above by $2dim(DCF)^2$, and the actual operation count can be much less when the matrix is sparse. We call this bound the *ODIF-full-bound.* The ODIF-full-bound is the best possible bound for the ODIF on the assumption that each block in the factorization of the DFT matrix is full. Python code written by Mike Hansen computed the ODIF-full-bound for $n = 1, ..., 13$. Figure 4.1 tabulates the operation count divided by $|S_n|$ for the ODIF-full-bound (ODIF$_{\text{bound}}$), an ODIF algorithm implemented by Jameson and Froehle (ODIF$_{JF}$), the theoretical bound on Clausen's DIT algorithm given in Clausen and Baum (1993) (C$_{\text{bound}}$), and Clausen't DIT implemented in Maslen and Rockmore (2000) (C$_R$).

n	$\text{ODIF}_{\text{bound}}$	ODIF_{JF}	C_{bound} [a]	C_R [b]	n^2
3	5.3	5.3	13	—	9
4	10.5	9.7	31	20	16
5	18.6	15.5	60	40	25
6	31.4	22.7	102.5	70	36
7	52.6	32.9	161	112	49
8	89.9	—	238	168	64
9	158.8	—	336	—	81
10	290.5	—	457.5	—	100
11	547.8	—	605	—	121
12	1060.7	—	781	—	144
13	2107.0	—	988	—	169

[a] Clausen and Baum (1993)
[b] Maslen and Rockmore (2000)

Table 4.1: Comparison of (operation count / $n!$).

While the ODIF-full bound eclipses the experimental version of Clausen's DIT for small n, it overestimates the operation count of the experimental ODIF. This indicates that the change of basis matrices for DCFs can be made quite sparse. Our current interest is finding the reason for the sparseness of these matrices. We are hoping that, by obtaining a formula for a specific intermediate basis at each step of the ODIF, we may obtain some critical information about this sparseness. I will conclude this document with the recent progress in developing a formula for the intermediate basis.

4.2 Introduction to the Tensor Basis

In this section I will briefly introduce the notion of the tensor basis. This concept is inspired by Mackey's Theorem, which establishes an isomorphism between a double coset space $\mathbb{C}S_k g S_h$ and $\mathbb{C}S_k \otimes_{gS_h g^{-1}} \mathbb{C}S_h$. Amazingly, from a doubly adapted Fourier basis of $\mathbb{C}S_k$ and a doubly adapted Fourier basis of $\mathbb{C}S_h$, one can create a spanning set that respects the $\mathbb{C}S_n$'s decomposition into the DCFs. Let us begin with Mackey's Theorem, which motivated the idea. We provide the proof presented in Weintraub (2003), because it is instructive in understanding the tensor basis.

Theorem 4.1. *(Mackey) Suppose G is a group and K, H are subgroups of G. Let A be a set of (H, K) double coset representatives, and let $H_g = gHg^{-1} \cap K$. If*

$\eta, \eta' \in H$ and $g\eta g^{-1} \in H_g$, let H_g to act on H from the left by $(g\eta g^{-1}) \bullet \eta' = \eta\eta'$. Then as a $(\mathbb{C}K, \mathbb{C}H)$ bimodule,

$$\mathbb{C}G \cong \bigoplus_{g \in A} \mathbb{C}(K) \otimes_{\mathbb{C}(H_g)} \mathbb{C}H.$$

Proof. We make a well-defined $(\mathbb{C}K, \mathbb{C}H)$-bimodule homomorphism from $\mathbb{C}G$ to $\mathbb{C}(K) \otimes_{\mathbb{C}(H_g)} \mathbb{C}H$. Let $g' \in G$. We can write $g' = \kappa g \eta$ for some $g \in A$, $\kappa \in K$, and $\eta \in H$. Then define the map

$$\phi : \mathbb{C}G \to \mathbb{C}K \otimes_{\mathbb{C}(H_g)} \mathbb{C}H$$

by $\phi(g') = \phi(\kappa g \eta) = \kappa \otimes_{\mathbb{C}(H_g)} \eta$. We claim that ϕ is well defined. Suppose $\overline{\kappa} g \overline{\eta} = \kappa g \eta$. Since $\kappa^{-1}\overline{\kappa} = g\eta\overline{\eta}^{-1}g^{-1}$, it follows that

$$\begin{aligned}
\overline{\kappa} \otimes_{\mathbb{C}(H_g)} \overline{\eta} &= \kappa(\kappa^{-1}\overline{\kappa}) \otimes_{\mathbb{C}(H_g)} \overline{\eta}\eta^{-1}\eta) \\
&= \kappa \otimes_{\mathbb{C}(H_g)} (\kappa^{-1}\overline{\kappa}) \bullet \overline{\eta}\eta^{-1}\eta \\
&= \kappa \otimes_{\mathbb{C}(H_g)} (g\eta\overline{\eta}^{-1}g^{-1}) \bullet \overline{\eta}\eta^{-1}\eta \\
&= \kappa \otimes_{\mathbb{C}(H_g)} \eta\overline{\eta}^{-1}\overline{\eta}\eta^{-1}\eta \\
&= \kappa \otimes_{\mathbb{C}(H_g)} \eta.
\end{aligned}$$

Thus the map is well defined. It is easy to see that ϕ is also a $(\mathbb{C}K, \mathbb{C}H)$-bimodule homomorphism from $\mathbb{C}G$ to $\mathbb{C}(K) \otimes_{\mathbb{C}(H_g)} \mathbb{C}H$.

Conversely, we will create a $(\mathbb{C}K, \mathbb{C}H)$-bimodule homomorphism from $\mathbb{C}(K) \otimes_{\mathbb{C}(H_g)} \mathbb{C}H$ to $\mathbb{C}G$. Consider the map $\sigma_g : \mathbb{C}K \times \mathbb{C}H \to \mathbb{C}G$ defined by $\sigma_g(\kappa, \eta) = \kappa g \eta$. If we establish that this map is H_g-balanced, then we can guarantee the existence of the unique $(\mathbb{C}K, \mathbb{C}H)$-bimodule homomorphism $\overline{\sigma_g}$ from $\mathbb{C}(H) \otimes_{\mathbb{C}(H_g)} \mathbb{C}(K)$ to $\mathbb{C}G$ such that $\sigma_g(\kappa, \eta) = \overline{\sigma_g}(\kappa, \eta)$ (Dummit and Foote, 1991). Let $g\eta g^{-1} \in H_g$. If $(\kappa', \eta') \in K \times H$,

$$\sigma_g(\kappa' g\eta g^{-1}, \eta') = \kappa' g\eta g^{-1} g\eta' = \kappa' g\eta\eta' = \sigma_g(\kappa, \eta\eta') = \sigma_g(\kappa, g\eta g^{-1} \bullet \eta)$$

Thus σ_g is H_g balanced, and there exists a unique $(\mathbb{C}K, \mathbb{C}H)$-bimodule homomorphism $\overline{\sigma_g}$ from $\mathbb{C}(H) \otimes_{\mathbb{C}(H_g)} \mathbb{C}(K)$ to $\mathbb{C}G$ such that $\sigma_g(\kappa, \eta) = \overline{\sigma_g}(\kappa, \eta)$. Define $\overline{\sigma} = \bigoplus_g \overline{\sigma_g}$. Then this is a $(\mathbb{C}K, \mathbb{C}H)$-bimodule homomorphism from $\bigoplus_{g \in A} \mathbb{C}(K) \otimes_{\mathbb{C}(H_g)} \mathbb{C}H$ to $\mathbb{C}G$ which maps $\kappa \otimes_{\mathbb{C}H_g} \eta$ to $\kappa g \eta$. Note that this map is the inverse of ϕ. This establishes the isomorphism. $\square$

It should also be noted that the above theorem does not depend on the choice of the double coset representative. Furthermore, by the way of the isomorphism ϕ, $\mathbb{C}KgH \cong \mathbb{C}(K) \otimes_{\mathbb{C}(H_g)} \mathbb{C}H$. The following claim is a direct corollary of Mackey's Theorem.

Theorem 4.2. *If $S_k g S_h$ is a double coset of type ℓ,*

$$\mathbb{C}S_k g S_h \cong \mathbb{C}S_k \otimes_{\mathbb{C}S_\ell} \mathbb{C}S_h.$$

Proof: Without loss of generality, let g be a canonical double coset representative. Then $gS_h g^{-1} \cap S_k = S_\ell$. □

At last, we introduce the notion of the tensor basis.

Theorem 4.3. *Suppose that H and K are subgroups of G and $G = \bigcup_{g \in A} KgH$, where A is a complete set of double-coset representatives. Further suppose B_K and B_H are doubly adapted bases for $1 = K_0 \leq \cdots \leq K_n = K$ and $1 = H_0 \leq \cdots \leq H_m = H$, respectively. Then*

$$B_g = \{b_1 g b_2 : b_1 \in B_K, b_2 \in B_H\}$$

is a spanning set of $\mathbb{C}KgH$ that is right-weakly-adapted to K's chain and left-weakly-adapted to H's chain.

Proof. B_g clearly spans $\mathbb{C}KgH$. Consider any $b_1 g b_2 \in B_g$. Let i be any integer between 0 and n, and let j be any integer between 0 and m. Suppose that e_i is the centrally primitive idempotent of $\mathbb{C}K_i$ corresponding to the $\mathbb{C}K_i$-isotypic space containing b_1, and that e_j is the centrally primitive idempotent of $\mathbb{C}H_j$ corresponding to the $\mathbb{C}H_j$-isotypic space containing b_2. Then clearly

$$e_i b_1 g b_2 e_j = b_1 g b_2,$$

because $e_i b_1 = b_1$ and $b_2 e_j$. This implies that $b_1 g b_2$ is contained in an $(\mathbb{C}K, \mathbb{C}H)$-isotypic space corresponding to e_i and e_j. □

Currently, we know a little bit more about this spanning set when $G = S_n$.

Theorem 4.4. *Let $S_k g S_h$ be a double coset of type ℓ, and let g be a canonical double coset representative. Denote the right $\mathbb{C}S_m$-isotypic space in $\mathbb{C}S_n$ corresponding to $r \vdash m$ by W_r, and the left $\mathbb{C}S_m$-isotypic space in $\mathbb{C}S_n$ corresponding to $l \vdash m$ by ${}_lW$. Suppose that $\{\alpha^{(m)}\}_{m=1}^{\ell}$ is a set of partitions such that $\alpha^{(m)} \vdash m$, and that b_1 is a vector in a doubly adapted basis of $\mathbb{C}S_k$ that is contained in $\bigcap_m^\ell W_{\alpha^{(m)}}$. Also, suppose that $\{\beta^{(m)}\}_{m=1}^{\ell}$ is a set of partitions such that $\beta^{(m)} \vdash m$, and that b_2 is a vector in a doubly adapted basis of $\mathbb{C}S_h$ that is contained in $\bigcap_m^\ell {}_{\beta^{(m)}}W$. If $\alpha^{(m)} \neq \beta^{(m)}$ for any m, then $b_1 g b_2 = 0$.*

Proof. Let $e_{\alpha^{(m)}}$ be the centrally primitive idempotent corresponding to $\alpha^{(m)}$. Then note that

$$b_1 \prod_{m=1}^{\ell} e_{\alpha^{(m)}} = b_1 \qquad \text{and} \qquad \prod_{m=1}^{\ell} e_{\beta^{(\ell+1-m)}} b_2 = b_2.$$

Recall that g commutes with S_ℓ. In particular, g commutes with $\prod_{m=1}^{\ell} \alpha^{(m)}$. Also, e_m commutes with e_j for any $j \leq m$. Now suppose that $a^{(m)} \neq b^{(m)}$ for some m. Then by orthogonality of idempotents,

$$\begin{aligned}
b_1 g b_2 &= \big(b_1 \prod_{m=1}^{\ell} e_{\alpha^{(m)}}\big) g \big(\prod_{m=1}^{\ell} \beta^{(\ell+1-m)} b_2\big) \\
&= b_1 g \big(\prod_{m=1}^{\ell} e_{\alpha^{(m)}}\big)\big(\prod_{m=1}^{\ell} e_{\beta^{(\ell+1-m)}} b_2\big) \\
&= b_1 g \big(\prod_{m=1}^{\ell} e_{\alpha^{(m)}} \prod_{m=1}^{\ell} e_{\beta^{(\ell+1-m)}}\big) b_2 \\
&= b_1 0 b_2 \\
&= 0.
\end{aligned}$$

□

Thus $b_1 g b_2 = 0$ unless the path corresponding to the frequency containing b_1 and the path corresponding to the frequency containing b_2 are the same up to the ℓth level of the Bratelli diagram. There is a strong evidence that the converse of this statement is true. We will address this matter in another paper.

Now suppose that $\phi : \mathbb{C}G \mapsto \mathbb{C}(K) \otimes_{\mathbb{C}(H_g)} \mathbb{C}H$ is Mackey's isomorphism. Then each $b_1 g b_2$ is a preimage of the simple tensor $b_1 \otimes_{\mathbb{C}(H_g)} b_2$. There is much evidence that nonzero elements in the spanning set of $\mathbb{C}S_n$ constructed in the manner of Theorem 4.3 forms a unique orthogonal basis of $\mathbb{C}S_n$. We call this basis the **tensor basis**.

4.2.1 Example

Consider S_3 as an (S_2, S_1) bimodule. The doubly adapted basis for S_2 is

- $(1) + (12)$
- $(1) - (12)$

For S_1, (1) is the doubly adapted basis. The canonical double coset representatives are $(1),(13),(23)$. Also $\mathbb{C}(S_{1g}) = \mathbb{C}$ clearly. Hence we can create the following spanning set of $\mathbb{C}S_3$:

- $\phi^{-1}(((1)+(12)) \otimes_{\mathbb{C}S_{1_{(1)}}} (1)) = (1)(1)(1) + (12)(1)(1) = (1) + (12)$
- $\phi^{-1}(((1)-(12)) \otimes_{\mathbb{C}S_{1_{(1)}}} (1)) = (1)(1)(1) - (12)(1)(1) = (1) - (12)$
- $\phi^{-1}(((1)+(12)) \otimes_{\mathbb{C}S_{1_{(13)}}} (1)) = (1)(13)(1) + (12)(13)(1) = (13) + (132)$
- $\phi^{-1}(((1)-(12)) \otimes_{\mathbb{C}S_{1_{(13)}}} (1)) = (1)(13)(1) - (12)(13)(1) = (13) - (132)$
- $\phi^{-1}(((1)+(12)) \otimes_{\mathbb{C}S_{1_{(23)}}} (1)) = (1)(23)(1) + (12)(23)(1) = (23) + (123)$
- $\phi^{-1}(((1)-(12)) \otimes_{\mathbb{C}S_{1_{(23)}}} (1)) = (1)(23)(1) - (12)(23)(1) = (23) - (123)$.

With the basis $\{(1),(12),(23),(123),(132),(13)\}$, this spanning set in the vector form is

$$\begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\-1\end{pmatrix}.$$

Next, consider S_3 as an (S_2, S_2) bimodule. There are two double cosets: $S_2(1)S_2$ and $S_2(23)S_2$. The canonical representatives of these double cosets are $(1),(23)$, respectively. Hence we can create the following spanning set of $\mathbb{C}S_3$:

- $\phi^{-1}(((1)+(12)) \otimes_{\mathbb{C}S_{2_{(1)}}} ((1)+(12))) = \phi^{-1}(2(1) \otimes_{\mathbb{C}S_2} ((1)+(12))) = (1) + (12)$
- $\phi^{-1}(((1)+(12)) \otimes_{\mathbb{C}S_{2_{(1)}}} ((1)-(12))) = \phi^{-1}((1) \otimes_{\mathbb{C}S_2} (1) - (12) + \otimes_{\mathbb{C}S_2} ((12)-(1))) = 0$
- $\phi^{-1}(((1)-(12)) \otimes_{\mathbb{C}S_{2_{(1)}}} ((1)+(12))) = \phi^{-1}((1) \otimes_{\mathbb{C}S_2} ((1)+(12)) + \otimes_{\mathbb{C}S_2} (-(12)-(1))) = 0$

- $\phi^{-1}(((1)-(12))\otimes_{\mathbb{C}S_{2(1)}}((1)-(12)))=\phi^{-1}(2(1)\otimes_{\mathbb{C}S_2}((1)-(12)))=2((1)-(12))$
- $\phi^{-1}(((1)+(12))\otimes_{\mathbb{C}S_{2(23)}}((1)+(12)))=\phi^{-1}(((1)+(12))\otimes_{\mathbb{C}}((1)+(12)))=(1)(23)(1)+(1)(23)(12)+(12)(23)(1)+(12)(23)(12)=(23)+(123)+(132)+(13)$
- $\phi^{-1}(((1)+(12))\otimes_{\mathbb{C}S_{2(23)}}((1)-(12)))=\phi^{-1}(((1)+(12))\otimes_{\mathbb{C}}((1)-(12)))=(1)(23)(1)-(1)(23)(12)+(12)(23)(1)-(12)(23)(12)=(23)+(123)-(132)-(13)$
- $\phi^{-1}(((1)-(12))\otimes_{\mathbb{C}S_{2(23)}}((1)+(12)))=\phi^{-1}(((1)-(12))\otimes_{\mathbb{C}}((1)+(12)))=(1)(23)(1)+(1)(23)(12)-(12)(23)(1)-(12)(23)(12)=(23)-(123)+(132)-(13)$
- $\phi^{-1}(((1)-(12))\otimes_{\mathbb{C}S_{2(23)}}((1)-(12)))=\phi^{-1}(((1)-(12))\otimes_{\mathbb{C}}((1)-(12)))=(1)(23)(1)-(1)(23)(12)-(12)(23)(1)+(12)(23)(12)=(23)-(123)-(132)+(13)$.

With the basis $\{(1),(12),(23),(123),(132),(13)\}$, the nonzero elements in this spanning set in vector form are

$$\begin{pmatrix}2\\2\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}2\\-2\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\-1\\1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\-1\\-1\\1\end{pmatrix}.$$

Note that these vectors are scalar-multiples of the vectors in the intermediate basis obtained in Section 2.1.4. They are also mutually orthogonal.

Expanding the ideas presented here, Mike Hansen and I have developed a conjecture for a very systematic method of determining the tensor basis using combinatorial objects called a *Young tableaux.* This conjecture has been confirmed for $n=1,\ldots,8$. Moreover, the ODIF implemented with the tensor basis has been almost as fast as the currently fastest DIF algorithm, which has been conjectured to run in $O(n^2|S_n|)$ time. Further details about this tensor basis will also be discussed in another paper.

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