Research Proposal:
Rigid Divisibility Sequences

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1 Introduction

Given a ring \( D \) and a polynomial \( f(x) \in D[x] \), one can define a sequence \( \{a_n\} \) in \( D \) by \( a_1 = f(0), a_2 = f(f(0)) \), and in general \( a_n = f(a_{n-1}) = f^n(0) \). When \( D \) is a Dedekind domain, we can ask about the prime factorization of the terms in the sequence. We say that the sequence \( \{a_n\} \) generated by \( f(x) \) as above is a rigid divisibility sequence if, for every prime ideal \( P \triangleleft D \), there is an integer \( p_k \) such that whenever the principal ideal \( (a_n) \) has \( P \) as a factor, the power of \( P \) in the prime factorization of \( (a_n) \) is exactly \( p_k \).

2 Proposed Research

I will explore the properties of rigid divisibility sequences. My main goal will be to give a complete characterization of which polynomials give rise to rigid divisibility sequences. Secondary goals include exploring other, similar conditions on these sequences, and seeing what the fact that the sequence \( \{f^n(0)\} \) is a rigid divisibility sequence can say about sequences where we let \( a_0 \) be arbitrary (rather than defining it to be equal to \( f(0) \)).

3 Prior Research

Similar to the notion of rigid divisibility sequence is that of strong divisibility sequence. In a strong divisibility sequence, it is the case that \( (a_n, a_m) = a_{(n,m)} \) for any \( n \) and \( m \). Thus, for instance, the Fibonacci numbers form a strong divisibility sequence. Complete characterizations (for instance \([1]\)) have been completed of certain classes of strong divisibility sequences, for instance those generated by second-order linear recurrences. However, all polynomials generate strong divisibility sequences in the manner described above, so that problem is uninteresting in this case. The notion of rigid divisibility sequence is a natural strengthening of the notion of a strong divisibility sequence, and one which is proving interesting.

There is very little on rigid divisibility sequences in the literature. The notion of a rigid divisibility sequence was (to my knowledge) first extensively
explored by me in [2]. Using a notion called “sequence factorization”, I was able to determine a large class of rigid divisibility sequences over \( \mathbb{Z} \), giving methods to generate polynomials which give rigid divisibility sequences, but I did not get close to a complete characterization. The results there suggested that looking at rigid divisibility sequences in general Dedekind domains might be fruitful for this, and they demonstrate how the notion of rigid divisibility sequence can be useful.

References
