Sperner's Lemma Implies Kakutani's Fixed Point Theorem

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Abstract

Kakutani's fixed point theorem has many applications in economics and game theory. One of its most well-known applications is in John Nash's paper [8], where the theorem is used to prove the existence of an equilibrium strategy in n-person games. Sperner's lemma, on the other hand, is a combinatorial result concerning the labelling of the vertices of simplices and their triangulations. It is known that Sperner's lemma is equivalent to a result called Brouwer's fixed point theorem, of which Kakutani's theorem is a generalization.

A natural question that arises is whether we can prove Kakutani's fixed point theorem directly using Sperner's lemma without going through Brouwer's theorem. The objective of this thesis to understand Kakutani's theorem, Sperner's lemma, and how they are related. In particular, I explore ways in which Sperner's lemma can be used to prove Kakutani's theorem and related results.

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To my parents, my sister, and my friends, thank you for being there.

Chapter 1

Introduction

In 1912, L. E. J. Brouwer proved a fixed point theorem for continuous functions from a compact, convex set in $\mathbb{R}^n$ to itself. Brouwer's fixed point theorem states that given a closed convex set $S \subset \mathbb{R}^n$ and a continuous function f from S to itself, there exists a point x in S such that $f(x) = x$. We call such point x a *fixed point* of f.

To give a simple illustration of Brouwer's fixed point theorem, consider the case when S is the closed interval $[0, 1] \subset \mathbb{R}$. Then, it is intuitively very easy to see that for any continuous function f from $[0, 1]$ to itself, the graph of $y = f(x)$ must either cross or touch the line $x = y$ at least once (see Figure 1.1).

There are many generalizations of Brouwer's fixed point theorem, including a fixed point theorem proved by Shizuo Kakutani in 1941 [7]. Kakutani's fixed point theorem considers upper semicontinuous, compact and convex-valued point-to-set mappings instead of continuous functions as in Brouwer's theorem. We define the terminology and discuss the concepts in greater detail in the next chapter.

In this thesis, I explore the relationship between Kakutani's theorem and a result from combinatorics called Sperner's lemma, which is a statement about the labelling of the vertices of a d-simplex (a d-dimensional triangle) and its subdivision into smaller simplices. (We call such subdivision a *triangulation*.)

In two dimensions, Sperner's lemma can be stated as follows: Consider a triangle (a 2-simplex) S whose three vertices are labelled 0, 1, and 2. Triangulate S into smaller triangles, such that any two small triangles are either disjoint, share a vertex, or share an edge. Label the new vertices resulting from this triangulation in any way such that if a vertex lies on the edge S

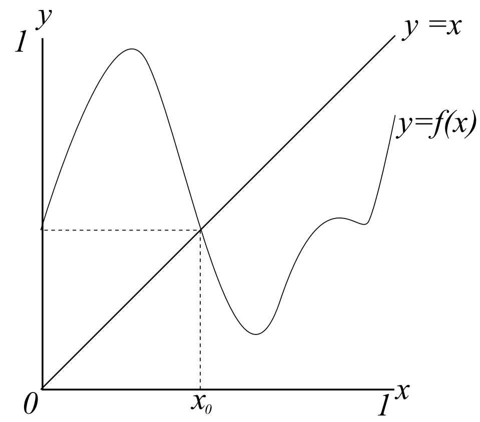

Figure 1.1: Illustration of Brouwer's fixed point theorem on $[0, 1]$

with endpoints labelled i and j (where $i, j \in \{0, 1, 2\}$ and are distinct), then this vertex may be labelled i or j only. If a vertex lies in the interior of S, it may receive any label. Then, Sperner's lemma states that there is an odd number of small triangles in the triangulation of S whose vertices receive three distinct labels 0, 1, and 2. We call such triangles *completely-labelled*. In particular, there is always at least one completely-labelled triangle. We illustrate this in Figure 1.2.

Sperner's lemma is equivalent to Brouwer's fixed point theorem as well as to a number of other related results, such as the Knaster-Kuratowski-Mazurkiewicz (KKM) lemma [6]. In this case, by equivalent, we mean that one result implies the other and is implied by the other without using additional results other than classical results of analysis such as convergence of a subsequence of any sequence of points in compact spaces.

The equivalence among the results mentioned above motivates the following question. Since Kakutani's fixed point theorem is a generalization of Brouwer's fixed point theorem, which is equivalent to Sperner's lemma, could we prove Kakutani's theorem directly from Sperner's lemma? In fact, this question also has practical significance. Algorithms for finding completely-labelled simplices, whose existence is guaranteed by Sperner's lemma, are very well developed. As we will see later, the proof of Brouwer's theorem from Sperner's lemma is constructive. Hence, a constructive proof of Kakutani's theorem directly from Sperner's lemma could provide insights into building efficient algorithms for finding fixed points of the point-to-set mappings specified in the hypothesis of Kakutani's theorem.

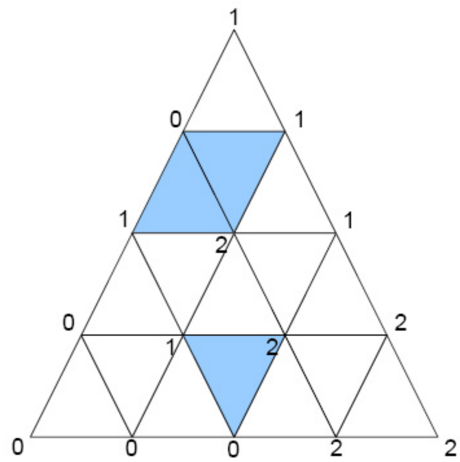

Figure 1.2: An illustration of Sperner's Lemma in 2 dimensions

In a paper published in 1980, D. I. A. Cohen presented a proof of Kakutani's theorem directly from Sperner's lemma without going through Brouwer's fixed point theorem [4], but this proof seems to be incomplete. Exactly how this proof is incomplete will be discussed in Section 2.4.1. The main result of this thesis is in Chapter 3, where we present a proof of Kakutani's fixed point theorem directly from Sperner's lemma. In this chapter, we also present preliminary approaches that we have considered for completing Cohen's proof. In Chapter 4, we also explored approaches for proving a related result called von Neumann's intersection lemma, which is known to be equivalent to Kakutani's theorem, using a polytopal generalization of Sperner's lemma in [6].

Chapter 2

Background

This chapter provides the necessary background information on Kakutani's fixed point theorem and Sperner's lemma. We first establish some basic terminology in Section 2.1. Section 2.2 provides background information on Kakutani's fixed point theorem, including the original proof using Brouwer's fixed point theorem. In Section 2.3, we state Sperner's lemma and show the equivalence between the lemma and Brouwer's fixed point theorem.

2.1 Terminology

In this section, we define some basic terminology that we will frequently use in the paper:

Graph For any function $\phi : X \to Y$, the graph of ϕ, denoted Gr (ϕ), is just given by our usual notion of a graph of a function. We formally state it as

$$\text{Gr }(\phi) = \{(x, y) \in X \times Y | y = f(x)\}.$$

Similarly, for any point-to-set mapping $f : X \to 2^Y$, we define the graph of f, denoted Gr (f) as

$$\text{Gr }(f) = \{(x, y) \in X \times Y | y \in f(x)\}.$$

Convex We call a set S *convex* if given any two points $x, y \in S$, the line segment between them is completely contained in S.

Convex combination Given a set of points $v^1, \ldots, v^d$, and scalars $\lambda_1, \ldots, \lambda_d$, such that

$$\sum_{i=1}^{d} \lambda_i = 1$$

and

$$\lambda_i \geq 0,$$

then the point $\sum_{i=1}^{d} \lambda_i v^i$ is called a *convex combination* of $v^1, \ldots, v^d$.

Affinely independent We say that a set of points $\{v^0, \ldots, v^d\} \subset \mathbb{R}^m$ is *affinely independent* if:

$$\sum_{i=0}^{d} \lambda_i v^i = 0$$

and

$$\sum_{i=0}^{d} \lambda_i = 0$$

implies that $\lambda_i = 0$ for all i.

Simplex A *(closed) d-simplex* is the set of convex combinations of an affinely independent set of $d+1$ points. In this paper, we are mostly concerned with closed simplices. Hence, any simplex mentioned is a closed simplex unless explicitly stated that it is not. To state it more formally, the simplex $(v^0 \ldots v^d)$ given by $\{v^0, \ldots, v^d\}$ is the set of convex combinations:

$$S = \left\{ \sum_{i=0}^{d} \lambda_i v^i | \lambda_i \geq 0 \text{ for all } i, \sum_{i=0}^{d} \lambda_i = 1 \right\}.$$

From here on, we let S denote a d-simplex specified by the points $v^0, \ldots, v^d$.

Vertices Given a d-simplex S, we call the points $v^0, \ldots, v^d$ the *vertices* of the simplex.

Face The convex combination of any subset of $\{v^0, \ldots, v^d\}$ is a simplex, which we call a *face* of the simplex $v^0 \ldots v^d$.

Barycentric coordinates Note that if y is a point in S, then we can write y as the following convex combination:

$$y = \sum_{i=0}^{d} \lambda_i v^i$$

for some $\lambda_0, \ldots, \lambda_d$. We say that the scalars $(\lambda_0, \ldots, \lambda_d)$ are the *barycentric coordinates* of y.

Carrier Suppose that for some $y \in S$, $y = \sum_{i=0}^{d} \lambda_i v^i$. If $\lambda_{i_j} > 0$ for $j = 0, \ldots, k$, then we say that the face $v^{i_0} \ldots v^{i_k}$ is the *carrier* of y. In other words, the carrier of y is the lowest-dimensional face of S containing y.

Triangulation A *triangulation* of S is a finite collection of simplices (and all their faces) $\{T_i\}$ such that

- $\cup T_i = S$ and
- for any distinct simplices T_j, T_k in the triangulation of S, $T_j \cap T_k$ is either empty, or a common face of each of them.

Mesh The *mesh* of a triangulation of S is the maximum diameter of a simplex in the triangulation.

2.2 Kakutani's Fixed Point Theorem

We now state Kakutani's fixed point theorem:

Theorem 2.1 (Kakutani's Fixed Point Theorem, 1941)**.** *Let S be a d-dimensional closed simplex and let f be an upper semicontinuous point-to-set mapping from S to nonempty, convex and compact subsets of S. Then, there exists a point $x \in S$ such that $x \in f(x)$.*

Kakutani's theorem applies only to point-to-set mappings that are convex and compact-valued, and in particular, *upper semicontinuous*. There are several different ways to state the definition of *upper semicontinuity*. We will discuss four statements of upper semicontinuity in the following subsection and establish that they are all equivalent to each other when restricted to point-to-set mappings to compact spaces. This is sufficient since Kakutani's theorem maps points in a convex, compact set S to convex, compact subsets of S.

2.2.1 Definitions of Upper Semicontinuity

We list four different statements of the definition of upper semicontinuity and present how the following definitions are related to each other.

Our first definition is the one used by Kakutani in [7].

Definition 2.2. A point-to-set map $f : X \to 2^Y$ is upper semicontinuous if $x_n \to x_0$, $y_n \to y_0$ (where $x_n, x_0 \in X$, $y_n \in \Phi(x_n) \subset Y$, and $y_0 \in Y$) implies that $y_0 \in f(x_0)$.

Any point-to-set mapping that satisfies the condition above has a graph that is a closed set. Hence, such mappings are also called closed mappings. In the case of Kakutani's theorem, the upper semicontinuity condition is equivalent to the condition that a mapping is closed. However, this is not true in general. In particular, the two conditions are equivalent in compact spaces but are distinct conditions in general. We shall discuss this further after we state all four statements of upper semicontinuity.

Before stating the next definition, note that there are some differences in notation and terminology in the literature. In this paper, the notation $f : X \to 2^Y$ for point-to-set mapping f means that f maps points in X to subsets of Y. However, there are several other different notations that are used in the literature. Our notation:

$$f : X \to 2^Y$$

or equivalently

$$f : X \to \mathcal{P}(Y),$$

where both 2^Y and $\mathcal{P}(Y)$ denote the power set of Y (the set of all subsets of Y) is commonly used. Another less commonly used notation is

$$f : X \multimap Y$$

where the symbol $\multimap$ denote a "multimap" where a point in X is mapped to possibly multiple points in Y. In [2], two right arrows are used to indicate that the mapping is a point to set mapping:

$$f : X \to\to Y.$$

We choose the notation $f : X \to 2^Y$ since this is the most commonly used and utilizes the most standard symbols.

Additionally, the term *hemicontinuity* has also been used in place of *semi*continuity, such as in [2]. This was done to avoid confusion with another meaning of semicontinuity that applies to point-to-point mappings. However, we will use the term semicontinuity, which is the more commonly used term, throughout this paper.

Now, we present the second definition of upper semicontinuity, as stated in [13, p.55]:

Definition 2.3. $f : X \to 2^Y$ is upper semicontinuous if:

1. $f(x)$ is compact for all $x \in X$, and
2. for any $x \in X$, given any $\epsilon > 0$, there exists a $\delta > 0$ such that if $z \in N_\delta(x) \cap X$, then $f(z) \subset N_\epsilon(f(x))$.

Here, $N_\epsilon(f(x))$ is a neighbourhood of "radius" ϵ of the set $f(x)$. For any set A, we define a neighbourhood of radius ϵ of a set as follows:

$$N_\epsilon(A) = \bigcup_{a \in A} N_\epsilon(a).$$

That is, it is the union of ϵ-neighbourhoods of all points in the set.

It is noted in [13] that if $\phi : X \to Y$ is a function and if we define $f(x) = \{\phi(x)\}$, then $f : X \to 2^Y$ is upper semicontinuous point-to-set mapping if and only if ϕ is a continuous function. Definition 2.3 implies Definition 2.2 (see [13, Lemma V.1.3]).

A third definition of upper semicontinuity comes from Border [2, Definition 11.3]. This definition is stated in terms of the upper (and lower inverses) of an open set E, which are defined as follows. If f is a point-to-set mapping, then the sets:

$$f^+(E) = \{x \in X | f(x) \subset E\}$$

and

$$f^-(E) = \{x \in X | f(x) \cap E \neq \emptyset\}$$

are the *upper inverse* and the *lower inverse* of E under f, respectively. Then, we define upper and lower semicontinuity:

Definition 2.4. $f : X \to 2^Y$ is upper semicontinuous if: for all $x \in X$, if x is in the upper inverse of an open set then so is a neighbourhood of x. Similarly, f is lower semicontinuous if whenever x is in the lower inverse of an open set, so is a neighbourhood of x.

Note that this definition is particularly nice since it explains the use of the term *upper* in upper semicontinuity, and analogously provides a definition of *lower* semicontinuity.

Finally, the following is the fourth definition of upper (and lower) semicontinuity in terms of upper (and lower) inverses. This is very similar to Definition 2.4 although worded slightly differently:

Definition 2.5. $f : X \to 2^Y$ is upper continuous if for each open set E in Y, the set $f^+(E)$ is open. f is lower semicontinuous if for each open set E, $f^-(E)$ is open.

It is not hard to see that Definitions 2.4 and 2.5 are in fact the same: Let us assume that f satisfies the hypothesis of Definition 2.4. Then, for any open $E \in Y$, if $x \in f^+(E)$, then there is a neighbourhood $N(x)$ of x such that $N(x) \subset f^+(E)$. By definition, $f^+(E)$ is open. Conversely, suppose we assume Definition 2.5's statement of upper semicontinuity. Since $f^+(E)$ is open, the condition of Definition 2.4 follows.

In the following, we show that Definitions 2.3 and 2.5 are equivalent if we assume that f maps to a compact space.

Proof. Let f be a point-to-set mapping from X to Y. We will first show that Definition 2.4 implies Definition 2.3.

Assume that f satisfies Definition 2.5. We wish to show that given any $\epsilon > 0$, we can find some $\delta > 0$ such that if $z \in N_\delta(x)$, then $f(z) \subset f(x)$, for all x in the domain.

Pick $\epsilon > 0$. Let $E = N_\epsilon(f(x))$. Since $f(x) \subset N_\epsilon(f(x))$, then $x \in f^+[E]$. Since E is an open set, by Definition 2.5, $f^+[E]$ is open. So, there must exist a $\delta > 0$ such that $N_\delta(x)$ is contained in $f^+[E]$. Definition 2.3 follows.

Now, we prove the other direction. We want to show that given any open set E in Y, the upper inverse of E under f is open.

Let E be a nonempty open set in Y. Let x be a point in X such that $f(x) \subset E$. Since E is open, there exists an $\epsilon > 0$ such that $N_\epsilon(f(x)) \subset E$. Pick such an ϵ. By Definition 2.3, there exists a $\delta > 0$ such that for all $z \in N_\delta(x)$, $f(z) \subset N_\epsilon(f(x))$. Repeat this for all $x \in X$ where $f(x) \subset E$. Let X_E denote the set of all $x \in X$ such that the above condition holds.

Let δ_x denote the δ that works for each x in X_U and let $K = \bigcup\limits_{\forall x \in X_E} N_{\delta_x}(x)$. Then, $K \subset X$ and K is open since it is a union of open sets. By the construction of K, if $x \in K$, then $f(x)$ is in E. So, $K = f^+[E]$ is open, and Definition 2.5 follows. □

This proof completes our earlier discussion about the equivalence of upper semicontinuous and closed mappings in compact spaces. Definition 2.3 of upper semicontinuity is in fact the condition for a mapping to be a closed mapping. In compact spaces, this definition is equivalent to the other definitions of upper semicontinuity, as we have just shown above. Hence, we may refer to the closedness condition as an upper semicontinuity condition, when the domain and codomain of our mappings with are compact spaces.

To show that the two conditions are not equivalent in general, consider the following examples which we take from [2].

Example 2.6 (Upper semicontinuous and closed mappings)**.** We define two point-to-set mappings on $\mathbb{R}$, which is not a compact space. First, we define the point-to-set mapping $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ as follows:

$$\gamma(x) = \begin{cases} \{1/x\} & \text{for } x \neq 0 \\ \{0\} & \text{for } x = 0. \end{cases}$$

It is easy to check that γ satisfies the condition stated in Definition 2.2, and hence it is a closed mapping. However, it is not upper semicontinuous since it is not continuous as a function.

Now, we define the point-to-set mapping $\mu : \mathbb{R} \to 2^{\mathbb{R}}$ by:

$$\mu(x) = (0, 1).$$

Note that μ is a constant map. Suppose E is an open subset of R. If E contains the open interval $(0, 1)$, then the upper inverse image of E under μ is the whole real line $\mathbb{R}$, which is open. Otherwise, if E does not contain $(0, 1)$, the upper inverse of E is the empty set, which is also open. So, μ is has an open upper inverse, and is upper semicontinuous by Definition 2.5. However, since $(0, 1)$ is open in $\mathbb{R}$, then the graph of μ in $\mathbb{R}^2$ is not closed. So, μ is not a closed mapping, and does not satisfy the condition in Definition 2.3.

2.2.2 Proof of Kakutani's Theorem from Brouwer's Fixed Point Theorem

The original proof of Kakutani's theorem by Shizuo Kakutani (1941) in [7] uses Brouwer's fixed point theorem, and it goes as follows.

Proof. We suppose that S is a d-dimensional simplex in $\mathbb{R}^d$ and $f : S \to 2^S$ is a convex and compact valued point-to-set mapping.

Let T^n be the n-th barycentric triangulation of S. We define a function (mapping a point to a point) $\phi_n : S \to S$ as follows: First, define ϕ on each vertex t of the triangulation T^n, by letting $\phi_n(t) = y$ for an arbitrary point y in $f(t)$. Then, the mapping ϕ_n defined on all vertices of T^n can be extended linearly to a continuous function ϕ_n from S to itself. That is, if s is not a vertex of T^n, then s is in some d-simplex of the triangulation (either interior or boundary), whose vertices we denote $w_1, \ldots w_d$. So, s is a convex combination of the w_j's. In other words, there exists nonnegative numbers s_i such that $\sum_{i=1}^{d} s_i = 1$ and

$$s = \sum_{i=1}^{d} w_i s_i.$$

Hence, we define $\phi_n(s)$ as:

$$\phi_n(s) = \sum_{i=1}^{d} \phi_n(w_i) s_i.$$

By Brouwer's fixed point theorem, ϕ_n has a fixed point. That is, there exists an $x_n \in S$ such that $x_n = \phi_(x_n)$. Hence, we obtain a sequence of fixed points of the linear extensions. By compactness of S, there exists a subsequence of $\{x_n\}$ which converges to some $x_0 \in S$. We claim that this is the fixed point of f.

To show this, let Δ_n be a d-dimensional simplex of T^n which contains x_n. Note that if x_n lies on a lower-dimensional simplex of T^n, then Δ_n is not unique. In this case, we can arbitrarily choose one of them. Let $x_n^0, x_n^1, \ldots, x_n^d$ be the vertices of Δ_n. Then, there exists a subsequence of $\{x_n^i\}$ that converges to x_0 for each $i \in \{0, 1, \ldots, d\}$. So, without loss of generality, we can assume that $\{x_n^i\}$ converges to x_0.

So, we have

$$x_n = \sum_{i=0}^{k} \lambda_n^i x_n^i$$

for suitable λ_n^i's ($i \in \{0, 1, \ldots, d\},\ n \in \{1, 2, \ldots\}$), with

$$\lambda_n^i \geq 0$$

and

$$\sum_{i=0}^{d} \lambda_n^i = 1.$$

Let y_n^i be a shorthand for $\phi(x_n^i)$. Then, $y_n^i \in f(x_n^i)$, and

$$x_n = \phi(x_n) = \sum_{i=0}^{d} \lambda_n^i \phi(x_n^i) = \sum_{i=0}^{d} \lambda_n^i y_n^i,$$

for each i.

Again, we use compactness to find a converging subsequence. In this case, we take a subsequence of the n's such that both $\{y_n^i\}$ and $\{\lambda_n^i\}$ converge to some y_0^i and λ_0^i, respectively. Then, we have

$$\lambda_0^i \geq 0,$$

$$\sum_{i=0}^{k} \lambda_n^i = 1,$$

and

$$x_0 = \sum_{i=0}^{d} \lambda_0^i y_0^i.$$

Since $x_n^i \to x_0$, $y_n^i \in f(x_n^i)$, and $y_n^i \to y_0^i$ for each $i \in \{0, 1, \ldots, d\}$, then by uppersemicontinuity of f, $y_0^i \in f(x_0)$, for each i. By the convexity of $f(x_0)$, then

$$\sum_{i=0}^{d} \lambda_0^i y_0^i \in f(x_0).$$

But $\sum_{i=0}^{d} \lambda_0^i y_0^i$ is just x_0 itself. So, $x_0 \in f(x_0)$. □

Now, we show that Kakutani's theorem holds not only for such point-to-set mappings on *simplices*, but also on *any bounded closed and convex sets* in a euclidean space.

Proof. Let $S \subset \mathbb{R}^d$ be a bounded closed and convex set on which an upper semicontinuous point-to-set mapping with convex and compact value is defined. Call this mapping f. Since S is bounded, we can find a closed d-simplex S' that contains S as a subset.

Let c be a point in the interior of S. We now define a point-to-set mapping $f' : S' \to 2^S$ as follows:

$$f'(x) = \begin{cases} f(x) & x \in S \\ \{c\} & x \notin S \end{cases}$$

We claim that f' is upper semicontinuous (for more detailed explanation, see [13, Theorem 1.4]). By Theorem 2.1, f' has a fixed point in S'. Since any point $x \in S'$
S is mapped to $c \in S$, then any fixed point of f' must be in S. This implies that f has a fixed point. □

Although Kakutani's theorem holds for bounded closed and convex sets in $\mathbb{R}^d$ in general, it is not a topological result like Brouwer's fixed point theorem which holds for any set that is homeomorphic to a closed ball. Nevertheless, there are generalizations of Kakutani's theorem that asserts the existence of fixed point on point-to-set mappings on nonconvex sets.

For instance, Theorem 3 in [12] states that if X is a subset in a locally convex Hausdorff topological vector space, and f is an upper semicontinuous point-to-set mapping from S to itself, with nonempty, convex and closed values, then for f to have a fixed point, it is necessary and sufficient that there exists a nonempty, compact and convex subset C of S such that $F(S) \cap C$ is nonempty.

2.3 Sperner's Lemma

We first define what *proper Sperner labelling* means. A labelling of the vertices of a triangulation of S is a *proper Sperner labelling*, if for each vertex t of a triangulation of S, t is labelled with the label of one of the vertices of the carrier of t. To state it more formally:

Definition 2.7. Let S be an d-simplex with vertices $v^0, \dots v^d$, where v^i is labelled i. Let T be a triangulation of S. Suppose that $(x_0, \dots, x_d)$ is the barycentric coordinates of $x \in S$. Let $L(x) = \{i \in \{0, \dots, d\} | \lambda_i > 0\}$ (that is, $L(x)$ is the set of labels such that x is carried by the vertices of S with labels in $L(x)$: $\{v^i | i \in L(x)\}$).

Then, the labelling of the vertices of T by $\{0, \dots, d\}$ is a *proper Sperner labelling* if for each $x \in T$, the label of x is one of the labels in $L(x)$.

We say that a simplex T_i in the triangulation T of the d-simplex S is *completely-labelled* if the vertices of T_i receives $d + 1$ distinct labels.

Having established the necessary terminology, we finally can state Sperner's Lemma:

Theorem 2.8 (Sperner's Lemma)**.** *Let T be a triangulation of the d-simplex S and each vertex $t \in T$ be given a proper Sperner labelling. Then, there are an odd number of completely-labelled simplices in the triangulation. (In particular, there is at least one completely-labelled simplex.)*

There are various ways to prove Sperner's Lemma. Here, we will present a proof that is similar to one in [10] and uses a "path-following approach".

Proof. We prove the lemma by induction on the dimension of the simplex. Consider $k = 1$. The triangulated 1-simplex is just a line segment that is divided into smaller intervals. Let us label the endpoints of the line (the vertices of the simplex) with 0 and 1, and other vertices (intervals' endpoints) in between with either 0 or 1. Note that if we trace the simplex from one endpoint to the other, we will change labels (from 0 to 1, vice versa) an odd number of times since the endpoints have distinct labels. The intervals in which we change label have endpoints of distinct labels, hence they are completely labelled.

Now that we have a base case, suppose that the theorem holds for all dimensions $d-1$ or below. We'll show that the theorem is true for a d-simplex S. We can think of the d-simplices in the simplex S as rooms with the $(d-1)$-faces as the walls. If a $(d-1)$-face of a simplex is itself a completely-labelled $(d-1)$-simplex with the vertices labelled $0, 1, \ldots, d-1$ (we pick this particular set arbitrarily, but any subset containing d labels will work), then we place a door on that wall, which connects the room to the adjacent room. These doors could lie either in the interior of S or on the boundary of S.

Since each $(d-1)$-face of S is itself a triangulated simplex, by the induction hypothesis we know that there is an odd number of doors on the boundary of S.

Now, observe that each room (each d-simplex in the triangulation) can have at most two doors, and if a room is completely labelled, it has exactly one door. So, if we enter through a door on the boundary, either we enter a completely-labelled simplex such that there is no other door connecting it to another room, or we enter a room with more than one door. Pick one of the other doors and continue to the next room. Since each room has at most two doors, then a room that has been passed through would not be visited again. Since the number of rooms is finite, then the procedure above must end, either by entering a room with no other door (which is a completely-labelled simplex) or exiting through a door on the boundary of S.

Since there are an odd number of doors on the boundary of S, then the paths above pair up only an even number of them. Then, there is an odd number of doors that lead to a completely-labelled simplex. There could be completely-labelled simplices that are not reachable through the boundary doors. However, these simplices must come in pairs, because their door must lead to a path that does not exit on the boundary. So, the total number of completely-labelled simplices must be odd. $\square$

2.3.1 Equivalence of Sperner's Lemma and Brouwer's Fixed Point Theorem

In this section, we present a proof of Brouwer's fixed point theorem from Sperner's lemma as well as a proof of Sperner's lemma using Brouwer's fixed point theorem. Although the relationship between these two results is not the main focus of this paper, these two proofs, combined with the proof of Kakutani's theorem from Brouwer's theorem, are useful in providing an intuition for approaching the main problem of proving Kakutani's theorem from Sperner's lemma.

Sperner's Lemma Implies Brouwer's Fixed Point Theorem

Here, we will prove Brouwer's theorem using Sperner's lemma. We consider a more specific case of Brouwer's fixed point theorem:

Theorem 2.9 (Brouwer's Fixed Point Theorem, 1910)**.** *Let S be a d-simplex in $\mathbb{R}^d$ and $f : S \to S$ a continuous function. Then, there exists some $x \in S$ such that $f(x) = x$.*

It is important to notice that Brouwer's theorem is actually much more general than the version that we state above. Namely, the theorem holds for any space S that is homeomorphic to a closed ball. That is, if a space can be continuously deformed into a closed ball or vice versa, then given any continuous function on this space, there is a point in the space that is fixed by it.

For our purpose, however, it is much more convenient to first prove the theorem for simplices, since this is the same space that we are concerned with in Sperner's lemma. The generalization to any space homeomorphic to a closed ball is not difficult, and will be briefly discussed after the following proof.

Proof. We assume that Sperner's lemma holds and we consider a d-simplex S and a continuous function f from S to itself. We wish to show that f has a fixed point in S.

Since S is a d-simplex, let us label its $d+1$ vertices with $0, 1, \ldots, d$. We denote the vertex of S that is labelled i as v^i Let T^δ denote a triangulation of S with mesh δ. We first choose a labelling rule for labelling each vertex in the triangulation as follows. Let t be a vertex of T^δ. Then, we may label t with i (where i is one of $0, \ldots, d$) if:

$$(t)_i \geq (f(t))_i$$

and

$$(t)_i > 0,$$

where $(x)_i$ denote the barycentric coordinate of x with respect to the vertex v^i of S, for any point x in S. Intuitively, we may label t with i if the image of t under f is further away (in the i^{th} barycentric coordinate) from the vertex v^i of S.

We can show that the labelling above is a proper Sperner labelling. First, if v^j is not a vertex of the carrier of t, then $(t)_j = 0$. So, t may not receive j as a label. Second, we need to show that each t can receive some label (no vertex is left without being able to receive a label). We know that

$$\sum_{i=0}^{d} v^i (t)_i = t,$$

$$\sum_{i=0}^{d} v^i (f(t))_i = f(t),$$

where $(t)_i \geq 0$ and $(f(t))_i \geq 0$ for all i, with

$$\sum_{i=0}^{d} (t)_i = 1$$

and

$$\sum_{i=0}^{d} (f(t))_i = 1.$$

In particular, there is some i such that $(t)_i$ is strictly greater than zero. Therefore, only consider the set of labels, call it L, where $(t)_i > 0$ for all $i \in L$. Hence, if $(t)_i < (f(t))_i$ for all values of i in L, we obtain

$$1 = \sum_{i \in L} (t)_i < \sum_{i \in L} (f(t))_i = 1,$$

which is a contradiction. Hence, for any $t \in S$, there is always some i that works. So, this is a proper Sperner labelling.

Let us take a sequence of triangulations of S, call it $\{T^\delta\}$, with the mesh δ converging to zero. Then, we label the vertices of each triangulation using the above method, such that Sperner's lemma guarantees the existence of a completely-labelled simplex in each T^δ. We shall denote this simplex by Δ_δ.

Let v_δ^i denote the vertex of Δ_δ that has the label i. For each i, we have the sequence of points $\{v_\delta^i\}$. Since S is compact, then there exists a converging subsequence, so without loss of generality, we can assume that $\{v_\delta^i\}$ converges to some point $x_i \in S$.

Since δ goes to zero, then in fact, for every i, the sequence $\{v_\delta^i\}$ converges to the same point. Hence, $x_0 = x_1 = \ldots = x_n$. So, for simplicity, we will call this limit point x_0.

We now show that the limit point x_0 can receive any of the $d+1$ possible labels. Suppose to the contrary, that x_0 cannot receive the label j. This means that $(x_0)_j < (f(x_0))_j$. Hence $d = (f(x_0))_j - (x_0)_j$ is strictly greater than 0. Now, choose $\epsilon = d/3$.

Recall that $x_n^j \to x_0$. Since f is continuous, then $f(x_n^j) \to f(x_0)$ also. These convergences also hold in each barycentric coordinate. So, there for the ϵ chosen above, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d((f(x_n^j))_j, (f(x_0))_j) < \epsilon$$

and

$$d((x_n^j)_j, (x_0)_j) < \epsilon.$$

This implies that $(x_n^j)_j < (f(x_n^j))_j$ for all $n \geq N$. This is a contradiction since it implies that x_n^j cannot receive the label j. Hence, x_0 may receive any of the $d+1$ possible labels.

This means that $(x_0)_i \geq (f(x_0))_i$ for all i. However, we know that

$$\sum_{i=0}^{d} (x_0)_i = \sum_{i=0}^{d} (f(x_0))_i = 1.$$

So, it must be the case that $(x_0)_i = (f(x_0))_i$ for all i, implying that $x_0 = f(x_0)$ as desired. $\square$

It is important to note that Brouwer's fixed point theorem is actually much more general than what we stated in Theorem 2.9. In Theorem 2.9, we state that any continuous function from a d-simplex S to itself must have a fixed point. Note however, that the theorem also holds when S is any convex set. A proof that extends the theorem for any convex set S can be found in [2, Corollary 6.6]. In fact, Brouwer's fixed point theorem is even more general. That is, it holds for any set S that is *homeomorphic* to a closed ball B. By homeomorphic to a closed ball, we mean that there exists a continuous bijection between S and B, such that we can continuously deform S into B and vice versa.

Brouwer's Fixed Point Theorem Implies Sperner's Lemma

Proof. Again, let S be a d-dimensional simplex with vertices $v^0, \ldots, v^d$, such that v^i is labelled i. Let T be any triangulation of S with vertices given a proper Sperner labelling. We wish to show that there exists a completely labelled simplex in T using Brouwer's fixed point theorem.

First, we define a function $f : S \to S$ as follows. For each vertex w in T, let $f(w) = v^j$, where j is the label of w. For any other point $u \in S$, u must lie in a simplex T_u in the triangulation T (note that T_u is not necessarily d-dimensional). So, u is a convex combination of the vertices of T_u, call them $t_0, \ldots, t_k$ (with $0 \leq k \leq d$), where k is the dimension of T_u. Hence, we extend f linearly:

$$f(u) = \sum_{i=0}^{k} u_i f(t_i).$$

It is easy to see that f is continuous in each simplex in T since it is linear. In fact, f is continuous in the entire domain S. To see this, consider two d-simplices T_1, T_2 in the triangulation T such that $T_1 \cap T_2 = T_3$, where T_3 is a common face of T_1 and T_2. Let $f|_{T_1}$ and $f|_{T_2}$ denote the function f restricted to T_1 and T_2, respectively. Then, for any $x \in T_3$, $f|_{T_1}(x) = f|_{T_2}(x)$. So, f is continuous on S.

Hence, by Brouwer's fixed point theorem, there is a point $x \in S$ that is fixed by f. Without loss of generality, assume that x is in the interior of S. We can make this assumption because if x is not in the interior of S, then there exists a maximum-dimensional proper face F of S that contains x in its interior. Then, we will be able to make exactly the same arguments as the following, with the only difference being the dimension of the simplex where x is an interior point of.

Let Δ denote a d-simplex in T that contains x, and let $w_0, \ldots, w_d$ denote the vertices of Δ. We can write x as a convex combination of these vertices:

$$x = \sum_{i=0}^{d} w_i x_i.$$

First, we show that x must lie in the interior of Δ. Note that since f is linear within Δ, then

$$\begin{aligned} f(x) &= f(\sum_{i=0}^{d} w_i x_i) && (2.1) \\ &= \sum_{i=0}^{d} f(w_i) x_i. && (2.2) \end{aligned}$$

Suppose to the contrary, that x lies in a proper face of Δ, then there is an index i where $x_i = 0$. Then, v^i is not a vertex of the carrier of $f(x)$, which means that $f(x) = x$ lies on a proper face of S. This contradicts our assumption that x is in the interior of S. So, we know that x must be in the interior of Δ. Hence, $x_i > 0$ for all i.

Now, we show that the images of $w_0, \ldots, w_d$ under f must be all the vertices of S, namely $\{v^0, \ldots, v^d\}$. Suppose to the contrary, that this is not true. Then, from Equation 2.2, we see that $f(x)$ lies on a proper face of S, which again is a contradiction.

Hence, the images of the vertices of Δ is all the vertices of S, which means that Δ is completely-labelled. This proves Sperner's lemma. □

2.4 Literature Review

In this section, we present a brief overview of some relevant existing results on Sperner's lemma and Kakutani's theorem. In particular, we present Cohen's approach to proving Kakutani's fixed point theorem directly from Sperner's lemma, followed by a generalization of Sperner's lemma to polytopes, which we will use in our proof of von Neumann's intersection lemma in Chapter 4.

2.4.1 Cohen's Paper

In this section, we reproduce Cohen's ideas for a proof of Kakutani's theorem from Sperner's lemma. Then, we present an example of how his argument does not quite work:

Cohen's "proof". Let S be an d-simplex with vertices labelled $0, 1, \ldots, d$. Let $f : S \to 2^S$ be a convex and compact-valued, upper semicontinuous point-to-set map from S to itself.

Let us assume that no fixed point exists. Let T be any triangulation of S. Let t be any vertex of T. There must exist a vertex w of S such that

1. w is a vertex of the carrier of t, and

2. there is a hyperplane dividing the simplex such that w and t are on one side and $f(t)$ is on the other.

Take any such w and label t with the label of w. If we apply this process to every vertex of T, we will obtain a labelled triangulation which is proper in the Sperner sense. Therefore, there will exist an d-simplex of T with a complete set of labels.

Taking finer and finer triangulations will produce smaller and smaller complete n-simplices in S and these will have at least one limit point x. Let m be any vertex of S. Let $v_1, v_2, \ldots$ be a sequence of vertices of the complete triangles all of which have the label m and which converge to x (by compactness of S).

Let x_m be the point of $f(x)$ closest to m. If x_m is 3ϵ closer to m than x is, then there is a $0 < \delta < \epsilon$ such that for all v_i within δ of x, $f(v_i)$ is within ϵ of $f(x)$ (by upper semicontinuity of f). But this means that $f(v_i)$ is closer to m than v_i is. This contradiction shows that $f(x)$ cannot be closer to m than x is.

But x cannot be closer (or as close) to each of the vertex S than $f(x)$ is. Hence a contradiction, showing that there has to be a point $x \in S$ such that $x \in f(x)$. □

However, this proof does not quite work. Consider the italicized portion of the proof above. The labelling condition that Cohen uses assigns labels based on whether there exists a certain hyperplane that separates the point and a vertex of the simplex from the image of the point. The condition does not imply any restriction on the distances between two of the points above. Hence, the contradiction in the italicized paragraph is not justified (see Figure 2.1).

We attempted to fix this proof by considering the same labelling function but using a different argument. In particular, we attempted to show that if x is the limit point of a sequence of completely labelled simplices in the triangulation of our simplex, then we should be able to assign to x any of the $d+1$ possible labels.

However, we find an example that shows that the labelling method itself is not strong enough to obtain the desired result, namely to show a fixed point of the mapping. The following is an example of a point x in a 2-simplex S that may receive all three labels but is not a fixed point of the point-to-set mapping:

Example 2.10. In Figure 2.2, we suppose that x is a point to which a subsequence of completely labelled simplices (labelled using Cohen's labelling function) converges to. This implies that there are hyperplanes that separates the vertices of these simplices and the vertex of the same label from the images if these vertices (which is contained in $N_\epsilon(f(x))$, by the upper semicontinuity of f. However, as illustrated in the figure, it is possible to have completely labelled simplices in a neighbourhood around x, such that $f(x)$ is far from x, and specifically, does not contain x.

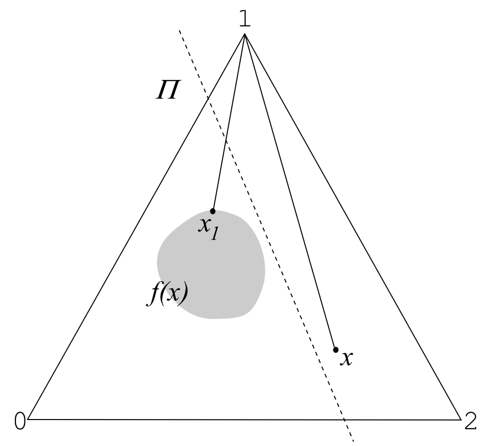

Figure 2.1: Here, we illustrate the italicized portion of Cohen's proof, for $m = 1$. The hyperplane (in this case, a line) Π separates x and the vertex of the simplex S with label 1 from $f(x)$. The point x_m in $f(x)$ is closer to the vertex labelled 1 than x is. It is easy to see that the remaining of the italicized portion of the proof fails to hold.

Hence, it is easy to see that if we have a point x that is the limit point of a sequence of completely labelled simplices with diminishing diameter, then x is not guaranteed to be a fixed point of the mapping.

In Chapter 3, I will describe some partial results as well as a complete proof of Kakutani's fixed point theorem using Sperner's lemma.

2.4.2 Generalizations of Sperner's Lemma

There are numerous generalizations of Sperner's lemma (such as [1, 3, 6] among others). One such generalizations that we will be using in this paper is a generalization of Sperner's lemma by De Loera, Peterson, and Su [6] to include the labelling of polytopes other than simplices.

Let P be a convex polytope in $\mathbb{R}^d$ with n vertices labelled $1, \dots, n$. (Note that $n \geq d + 1$.) We shall call such polytope as an (n, d)-polytope. A triangulation of P is a subdivision of P into simplices as we defined previously in the context of the original Sperner's lemma. We then consider a labelling of the vertices of a triangulation of P. We say that a d-simplex in a triangulation is *fully-labelled* or is a *full cell* is its labels are all distinct.

The following is the statement of their main result, which we shall refer

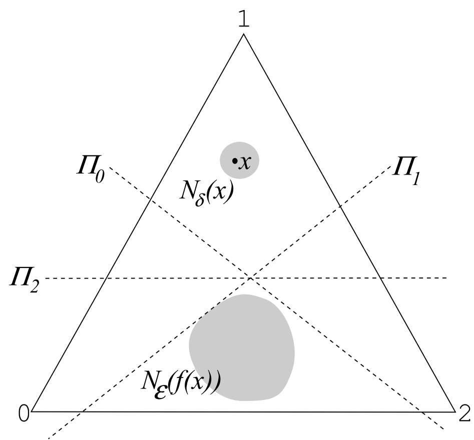

Figure 2.2: An example where Cohen's labelling does not guarantee that the limit point of the completely-labelled simplices is a fixed point.

to by the *polytopal Sperner lemma*.

Theorem 2.11 (De Loera, Peterson, Su, 2002)**.** *Any Sperner labelling of a triangulation of an (n, d)-polytope P must contain at least $n - d$ full cells.*

Additionally, the following result [6, Proposition 3] asserts that the full cells in Theorem 2.11 form a "cover" of P.

Proposition 2.12 (De Loera, Peterson, Su, 2002)**.** *Let P be an (n, d)-polytope with Sperner-labelled triangulation T, and let $f : P \to P$ be a piecewise linear function mapping each vertex of T to the vertex of P that shares the same label. Note that f is then linear on each d-simplex in T. Then, the map f is surjective, and thus the collection of full cells in T forms a cover of P under f.*

Therefore, a Sperner labelling of the vertices of an (n, d)-polytope not only guarantees the existence of $n - d$ full cells, but guarantees that each n labels appear in a full cell, and more specifically, that the full cells form a cover under the linear mapping that maps their vertices to the vertices of P of the same labels. The following example illustrates the theorem and proposition above.

Now, we apply the map f defined in Proposition 2.12 to the full cells $T_1, \ldots, T_4$ above (see Figure 2.4). The image of the full cells, under the function f defined in Proposition 2.12, forms a cover of the P.

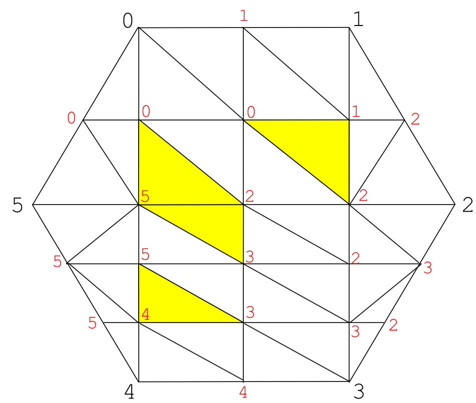

Figure 2.3: An illustration of the polytopal Sperner's lemma (Theorem 2.11 for a triangulation of a $(6, 2)$-polytope.

Example 2.13. Consider the $(6, 2)$-polytope illustrated in Figure 2.3. This polytope, which we will denote P, is a hexagon with vertices labelled 0, 1, $\ldots, 5$. We triangulate P and give a proper Sperner labelling to the vertices of the triangulation. As guaranteed by Theorem 2.11, we see that there are $n - d = 6 - 2 = 4$ full cells in this triangulation of P, namely $T_1 = (0, 1, 2)$, $T_2 = (0, 2, 5)$, $T_3 = (2, 3, 5)$, and $T_4 = (3, 4, 5)$.

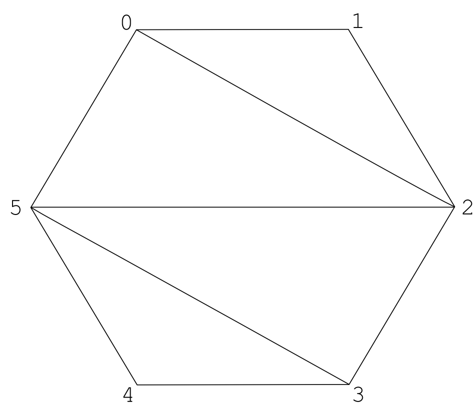

Figure 2.4: The image of the full cells in Figure 2.3 under the function f in Proposition 2.12 covers P

Chapter 3

Sperner's Lemma and Kakutani's Fixed Point Theorem

In this chapter, we present our main result, a proof of Kakutani's theorem directly from Sperner's lemma. We first present a proof of a weaker version of Kakutani's theorem from Sperner's lemma in Section 3.1. Then in Section 3.2, we describe the various preliminary approaches that we have taken to prove Kakutani's theorem from Sperner's lemma and why they did not work. Finally, in Section 3.3, we present a complete proof of Kakutani's theorem directly from Sperner's lemma.

3.1 Sperner's Lemma Implies Weak Kakutani's Theorem

Proving Kakutani's theorem from Sperner's lemma turned out to be quite difficult because the upper-semicontinuity condition did not seem to be strong enough. In this section, we will first prove a weaker version of Kakutani's fixed point theorem from Sperner's Lemma, where we require the point-to-set mapping f to be *uniformly upper semicontinuous*, which is a stronger condition than upper semicontinuous.

Definition 3.1 (Uniform upper semicontinuity)**.** We say that f is *uniformly upper semicontinuous*, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any points x, y in the domain such that $d(x, y) < \delta$, then $f(x) \subset N_\epsilon(f(y))$ (and $f(y) \subset N_\epsilon(f(x))$).

Note that the uniform upper semicontinuity condition as defined above is equivalent to requiring that if two points x, y are close with respect to the metric on the domain (in this case, the usual Euclidean metric), then the images are close to each other, in the sense that one is contained in an ϵ neighbourhood of the other, and vice versa. In fact, this notion of distance between sets is called the the *Hausdorff metric*, which we shall define more precisely later when we use it in the proof of the following theorem.

Theorem 3.2 (Weak Kakutani Fixed Point Theorem)**.** *Let S be an d-dimensional closed simplex and let f be a uniformly upper semicontinuous point-to-set mapping from S to convex and compact subsets of S. Then, there exists a point $x \in S$ such that $x \in f(x)$.*

Note that since uniform upper semicontinuity is a stronger condition than just upper semicontinuity, then the modified Kakutani fixed point theorem is a weaker theorem than the original Kakutani fixed point theorem.

Proposition 3.3. *Sperner's lemma implies the weak Kakutani's fixed point theorem.*

Proof. Let S be an d-dimensional (closed) simplex with vertices $v^0, \ldots, v^d$, where the label the vertex v^i with i. Let f be a uniformly upper semicontinuous point-to-set mapping from S to compact and convex subsets of S. We wish to show that there is a point $x \in S$ such that $x \in f(x)$.

We begin by stating a labelling function for points in S. To do this, we first construct d-dimensional simplices in S as follows: We define I_i to be the set

$$\{v^0, \ldots, v^d\} \backslash \{v^i\}$$

and we define $S_i(t)$ to be the simplex formed by the convex hull of t and the elements of I_i. If t is not contained in the convex hull of I_i, then $S_i(t)$ is an d-dimensional simplex contained in S. On the other hand, if t is contained in the convex hull of I_i, $S_i(t)$ will be a simplex in S of a lower dimension. We will only consider the simplices $S_i(t)$'s that are d-dimensional. Specifically, they are all $S_i(t)$ such that i is a vertex of the carrier of t. So, if t lies on an l-dimensional face of S with $0 \leq l \leq d$ (equivalently, if the carrier of t is l-dimensional), then we have $l + 1$ many of such $S_i(t)$'s.

Example 3.4. Figure 3.1 illustrates the d-dimensional simplices whose union is S when $d = 2$. The leftmost figure is the case when $l = d = 2$, the middle figure is when $l = 1$, and the rightmost is when $l = 0$.

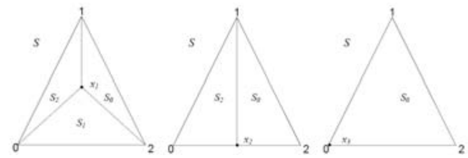

Figure 3.1: The construction of the simplices $S_i(t)$'s, when S is a 2-dimensional simplex.

Let $h(t)$ denote the point in $f(t)$ that is closest to t. Note that $h(t)$ is unique since $f(t)$ is convex. Then, we may label t with i if $h(t) \in S_i(t)$.

[Note: To see that $h(t)$ is unique if $f(t)$ is convex, we suppose to the contrary, that there exists two distinct points h_1, h_2 that are closest to x. Then consider the line segment connecting the two points. Since the $f(x)$ is convex, it contains the line segment. It is easy to see that the segment contains a point that is closer to x than h_1, h_2 are, which is a contradiction.]

We now show that this is a proper Sperner labelling by showing that all points in S can receive a label, and that the label of t is one of the labels of the vertices of its carrier. Let carr (t) denote the set of vertices of the carrier of t. If the carrier of t is k-dimensional, then there are $k+1$ many d-dimensional simplices we obtain from the procedure above and they are the simplices $S_i(t)$ such that $i \in$ carr (t). Next, we see that for all values of d, the simplices $S_i(t)$ form a cover for S, so $h(t)$ must lie in $S_i(t)$ for some $i \in$ carr t. Hence, each point in S can receive a label (additionally, note that if $h(t)$ lies in the boundary between several $S_i(t)$'s, t may receive one of several labels. Else, there is only one i that we can use to label t). Therefore, this labelling is a proper Sperner labelling.

Now, suppose to the contrary that for each point $t \in S$, $t \notin f(t)$. Then, let us triangulate S and label each vertex t of the triangulation described above. By Sperner's lemma, there exists a completely labelled simplex in the triangulation.

Take a sequence of finer and finer triangulations of S such that we obtain an infinite sequence of completely labelled simplices with mesh tending to zero. By the compactness of the domain, there is a subsequence of completely labelled d-simplices, call it Δ_n, converging to a point $x \in S$. Let us denote the vertex of Δ_n that is labelled i by v_n^i.

Here, we use the Hausdorff metric to specify distances between the any pair of sets $S_i(x)$, which is defined as follows.

Definition 3.5 (Hausdorff metric)**.** Let X, Y be compact sets of a metric space M. The Hausdorff distance between X and Y, denoted $d_H(X,Y)$, is the minimal number r such that any closed neighbourhood of radius r around each $x \in X$ contains a point $y \in Y$, vice versa. More precisely, we define $d(X,Y)$ as follows:

$$d_H(X,Y) = \max\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\},$$

where $d(x,y)$ is the metric on M.

Note that we can equivalently formulate the Hausdorff metric as follows:

$$d_H(X,Y) = \inf\{r \in \mathbb{R}_{\geq 0} | X \subset N_r(B), Y \subset N_r(X)\}.$$

Since the sets $S_i(x)$ are closed and bounded subsets of the Euclidean space, then they are compact, and we can use the Hausdorff metric above to find the distance between any two such sets. In fact, we claim that since $v_n^i \to x$ for each i, then

$$S_j(v_n^i) \to S_j(x)$$

for all i, j, where the convergence is in the Hausdorff sense. That is, given any $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, then

$$d_H(S_j(v_n^i), S_j(x)) < \delta.$$

We show this claim as follows. By the convergence of $\{v_n^i\}$ to x, given any $\delta > 0$, there exists $M_\delta \in \mathbb{N}$ such that for all $n \geq M_\delta$,

$$v_n^i \in N_\delta(x).$$

Recall that $S_j^n(v_n^i)$ is the convex hull of $I_j \cup \{v_n^i\}$. Also recall that $S_j(x)$ is the convex hull of $I_j \cup \{x\}$. So, the d-simplices $S_j(v_n^i)$ and $S_j(x)$ have d vertices in common, out of the $d+1$ vertices that each of them has.

Then, consider a point y in $S_j(v_n^i)$. We can write y as a convex combination of the vertices of $S_j(v_n^i)$, namely the $d+1$ points in $I_j \cup \{v_n^i\}$:

$$y = \sum_{w^k \in I_j \cup \{v_n^i\}} (y)_k \, w^k,$$

where each $(y)_k$ is nonnegative and $\sum_{\forall k} (y)_k = 1$.

Let w^0 denote the vertex v_n^i of $S_j(v_n^i)$. So, $w^1, \ldots, w^d$ are in I_j and are also vertices of $S_j(x)$. Then, we can rewrite y as follows:

$$y = (y)_0 \, v_n^i + \sum_{w^k \in I_j} (y)_k \, w^k.$$

Then, the point z given by changing v_n^i above to x gives us a convex combination of vertices of $S_j(x)$:

$$z = (y)_0\, x + \sum_{w^k \in I_j} (y)_k\, w^k.$$

If we order the vertices of $S_j(x)$ in the same order as the way we order the vertices of $S_j(v_n^i)$ above, then the point z has the same barycentric coordinate as y, but with respect to a *slightly* different set of $d+1$ points (i.e. same vertices except for v_n^i and x).

Hence, using the Pythagorean theorem, the distance between y and z is bounded above by the sum of their distance in each barycentric coordinate:

$$\begin{aligned} d(y,z) &< d((y)_0\, v_n^i, (y)_0\, x) + \sum_{k=1}^{d} d((y)_k\, w^k, (y)_k\, w^k) \\ &= d((y)_0\, v_n^i, (y)_0\, x) \\ &= (y)_0\, d(v_n^i, x) \\ &< (y)_0\, \delta < \delta \end{aligned}$$

This shows that for each $y \in S_j(v_n^i)$ (with $n \geq M_\delta$),

$$\inf_{z \in S_j(x)} d(y,z) < \delta.$$

Similarly, it also shows that for each $z \in S_j(x)$,

$$\inf_{y \in S_j(v_n^i)} d(y,z) < \delta.$$

So for all $n \geq M_\delta$, we have

$$d_H(S_j(v_n^i), S_j(x)) = \max\{ \sup_{y \in S_j(v_n^i)} \inf_{z \in S_j(x)} d(y,z), \sup_{z \in S_j(x)} \inf_{y \in S_j(v_n^i)} d(y,z)\} < \delta,$$

which proves the convergence of $\{S_j(v_n^i)\}_{n \in N}$ to $S_j(x)$, with respect to the Hausdorff metric.

In particular, note that this convergence holds for $j = i$:

$$S_i(v_n^i) \to S_i(x).$$

At this point, let us remind ourselves that we defined the sets $S_j(a)$ for any points a in the simplex S in order to decide what label we should assign to the point a. Specifically, if $h(a)$, the point in $f(a)$ that is closest to a is contained in $S_j(a)$, then we may assign to a the label j.

Having recalled this, to show that $h(v_n^i)$ converges to $h(x)$. First, note that the convergence of $S_i(v_n^i)$ to $S_i(x)$ implies that there exists $M \in \mathbb{N}$ such that for all $n \geq M$, we have

$$h(v_n^i) \in S_i(x).$$

Fix $\epsilon > 0$. By the uniform upper semicontinuity of f, there exists $0 < \delta < \epsilon$ such that if $d(v_n^i, x) < \delta$, then

$$f(v_n^i) \subset N_\epsilon(f(x)) \text{ and}$$

$$f(x) \subset N_\epsilon(f(v_n^i)).$$

Thus, as $\epsilon \to 0$, then the sets $\{f(v_n^i)\}$ converges to $f(x)$ with respect to the Hausdorff metric.

We claim that this implies that $\{h(v_n^i)\}$ converges to $h(x)$: Let k_n^i be the point in $f(v_n^i)$ closest to x. Since $f(v_n^i) \to f(x)$, then the triangle inequality gives

$$k_n^i \to h(x).$$

Since $v_n^i \to x$, then

$$d(v_n^i, k_n^i) \to d(x, k_n^i).$$

Let h_0 denote the limit of $\{h(v_n^i)\}$. (Since $h(v_n^i) \in f(v_n^i)$, $f(v_n^i)$ converges to $f(x)$, and f is closed valued, then h_0 is in $f(x)$.) Since $d(v_n^i, k_n^i) \geq d(v_n^i, h(v_n^i))$ (by the definition of $h(v_n^i)$), then in the limit,

$$d(x, k_n^i) \geq d(x, h_0).$$

If k_0 is the limit of k_n^i, then

$$d(x, h_0) \geq d(x, k_0).$$

So, in the limit,

$$d(x, h_0) = d(x, k_0).$$

Since $k_0 = h(x)$ and $h(x)$ is unique, then $h_0 = h(x)$. So,

$$h(v_n^i) \to h(x).$$

Using the above convergence, we show that f has a fixed point. Since $S_i(x)$ is closed and

$$h(v_n^i) \in S_i(x)$$

for all $n \geq M$, then their limit point, $h(x)$, must also be in $S_i(x)$. Since this is true for all values of i, $h(x)$ is contained in all of $S_i(x)$. However, by the construction of the $S_i(x)$'s, the only point that belong to all of the $S_i(x)$'s is x itself. So, $x = h(x) \in f(x)$. Thus, f has a fixed point. □

3.2 Preliminary Approaches

Here, we describe the preliminary approaches that we have taken in our attempt to prove Kakutani's fixed point theorem directly from Sperner's lemma. The various methods that we have tried have the same basic idea, with the main difference among them to be the labelling methods that we use.

We first outline the basic idea of our argument:

1. Choose a specific method for labelling points in S such that this is a proper Sperner labelling.

2. Consider an infinite sequence of finer and finer triangulations of S, and label the vertices in each triangulation using the method in chosen above.

3. By Sperner's lemma, there exists a completely-labelled simplex in each triangulation. Hence, we obtain a sequence of smaller and smaller completely-labelled simplices.

4. Since S is compact, there is a converging subsequence. Let us call the limit point x_0.

5. Show that x_0 can be given any of the $d + 1$ possible labels.

6. Using the properties of the labelling, show that x_0 is the desired fixed point.

Note that in step 5, how we show that the limit point x_0 can be given any of the $d + 1$ possible labels depend on the labelling method that we choose in step 1. Similarly, how we show step 6 also depends on the labelling method chosen in step 1.

3.2.1 Labelling Methods

Let S be a d-simplex with vertices $v^0, \ldots, v^d$, where the label the vertex v^i with i. Let f be a uniformly upper semicontinuous point-to-set mapping from S to compact and convex subsets of S.

Method 1 This is the labelling method that we use in the proof of the weak Kakutani fixed point theorem above. We restate the method:

We define I_i to be the set

$$\{v^0, \dots, v^d\} \setminus \{v^i\}$$

and we define $S_i(t)$ to be the simplex formed by the convex hull of $I_i \cup \{t\}$. If t is not contained in the convex hull of I_i, then $S_i(t)$ is an d-dimensional simplex contained in S. On the other hand, if t is contained in the convex hull of I_i, $S_i(t)$ will be a simplex in S of a lower dimension. We will only consider the simplices $S_i(t)$'s that are d-dimensional. Specifically, they are all $S_i(t)$ such that i is a vertex of the carrier of t. So, if t lies on an l-dimensional face of S with $0 \leq l \leq d$ (equivalently, if the carrier of t is l-dimensional), then we have $l + 1$ many of such $S_i(t)$'s.

Let $h(t)$ denote the point in $f(t)$ that is closest to t. Note that $h(t)$ is unique since $f(t)$ is convex. Then, we may label t with i if $h(t) \in S_i(t)$.

Method 2 In this method, we label a point in S based on the barycentric coordinates of the point relative to the barycentric coordinate of a point in its image.

For all $x \in S$, let $h(x)$ denote the point in $f(x)$ closest to x. Again, we note that $h(x)$ is unique since $f(x)$ is convex. Let $(x)_i$ denote the i^{th} barycentric coordinate of x, then we may give a label i to x if:

- v^i is a vertex of the carrier of x (i.e., if $(x)_i > 0$) and
- $(x)_i \geq (h(x))_i$.

We have shown in the previous section that Method 1 is a proper Sperner labelling. Hence, we now show that Method 2 is also a proper Sperner labelling:

Proof. Let $((x)_0, (x)_1, \dots, (x)_k)$ be the barycentric coordinate of x, and $((h(x))_0, (h(x))_1, \dots, (h(x))_d)$ be the barycentric coordinates of $h(x)$. Then, $(x)_i, (h(x))_i \geq 0$ for all i, and

$$\sum_{i=0}^{d} (x)_i = 1,$$

$$\sum_{i=0}^{d} (h(x))_i = 1.$$

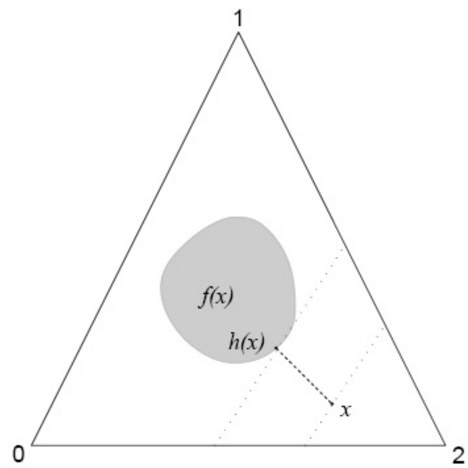

Figure 3.2: An illustration of labelling using Method 2

Let $L(x) = \{i|(x)_i > 0,\ i \in \{0, \dots, d\}\}$. That is, $L(x)$ is the set of labels that carries x. Then, $\sum_{i \in L(x)} (x)_i = 1$.

Since the carriers of x and $h(x)$ might not be the same, then $\sum_{i \in L(x)} (h(x))_i \leq 1 = \sum_{i \in L(x)} (x)_i$. So, there must exists some j such that $(h(x))_j \leq (x)_j$, for some $j \in L(x)$. Therefore, each $x \in S$ can receive a label. Since the labelling require that the label of x is one of the labels if the vertices that carry x, this is a proper Sperner labelling. □

3.2.2 Problems with Methods 1 and 2

Methods 1 and 2 are two examples of the labelling methods that are not strong enough to help us prove Kakutani's theorem from Sperner's lemma. Here, we explain why the two labelling methods fail.

Recall the basic idea of our proof of Kakutani's theorem from Sperner's lemma, which we presented earlier in this chapter. In step 5, we use the properties of our labelling methods to assert that x_0, the limit point of our converging subsequence of simplices, may receive any the possible $d+1$ labels. Then in step 6, we use the fact that x_0 may receive any of the labels to show that it has to be contained in its own image, hence showing that it is the desired fixed point.

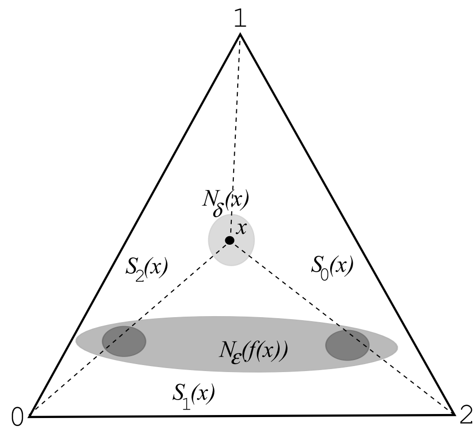

Figure 3.3: An illustration of Example 3.6

Method 1 fails in step 6. That is, it is possible to find a limit point x_0 that may receive any of the labels, but is not a fixed point of the mapping.

Example 3.6. See Figure 3.6. In this example, the point x is the limit point of a sequence of points (not pictured). The tail of this sequence is contained in $N_\delta(x)$. Since f is upper semicontinuous, the image of points in $N_\delta(x)$ is contained in $N_\epsilon(f(x))$, for some $\epsilon > 0$, $\delta > 0$. The figure illustrates that it is possible that x is the limit point of a sequence of points which could receive the labels 0, 1, or 2 (note that the two circles contained in $N_\epsilon(f(x))$ intersects $S_1(x)$, $S_2(x)$, and $S_3(x)$), but x is not contained in its own image.

On the other hand, method 2 fails in step 5. That is, x_0 could be the limit point of a sequence of points with label i, but x_0 may not receive the label i. Example 3.7 illustrates this.

Example 3.7. See Figure 3.7. The point x in S is the limit point of a sequence of points (here represented by z). As illustrated, the image of each z lies in an ϵ neighbourhood of the image of x, since f is upper semicontinuous. Note that the points z may receive the label 1, since each z is closer to 1 than $h(z)$ is in barycentric distance ($h(z)$ is not pictured, but it is clear from the figure since the entire image $f(z)$ is further away from the vertex labelled 1 than z is). However, x may not be labelled 1 because $h(x)$ is closer to 1 compared to x.

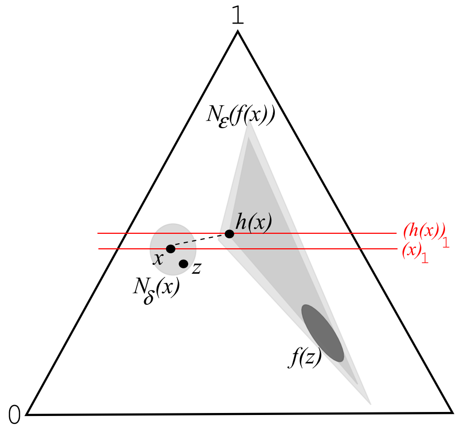

Figure 3.4: An illustration of Example 3.7

3.3 Sperner's Lemma Implies Kakutani's Fixed Point Theorem

In this section, we present a proof that Sperner's lemma implies Kakutani fixed point theorem. Throughout the section, we let S be a d-dimensional (closed) simplex with vertices $v^0, \ldots, v^d$, where we label the vertex v^i with the label i. Let f be an upper semicontinuous point-to-set mapping from S to compact and convex subsets of S. We wish to show that there is a point $x \in S$ such that $x \in f(x)$.

In addition to Sperner's lemma, we use the following result:

Lemma 3.8 (von Neumann's Approximation Lemma, 1937)**.** *Let $\gamma : E \to 2^F$ be an upper semicontinuous point-to-set mapping with nonempty compact convex values, where $E \subset \mathbb{R}^m$ is compact and $F \subset \mathbb{R}^k$ is convex. Then for any $\epsilon > 0$ there is a continuous function f such that $Gr\,(f) \subset N_\epsilon(Gr\,(\gamma))$.*

This lemma asserts that given any upper semicontinuous point-to-set mapping from a compact space to a convex space, we can find a continuous function that is "close" to it. Figure 3.3 illustrates this. Although we will not prove this lemma here, a proof can be found in Appendix A.

Here is an outline of our argument. We begin by triangulating the d-simplex S. We choose a rule for labelling each vertex of the triangulation and show that it is a proper Sperner labelling. (In this case, we use the above lemma in our choice of labels.) By Sperner's lemma, using such labelling on any triangulation of S, there exists a completely labelled simplex

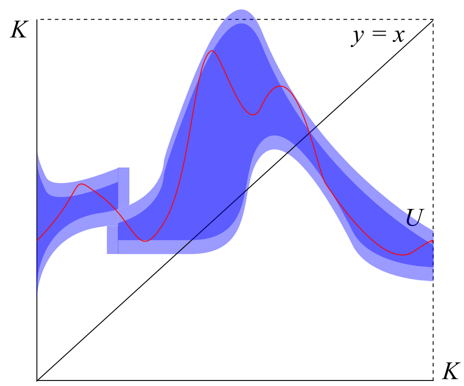

Figure 3.5: An illustration for Lemma 3.8, for a point-to-set mapping $f : K \to 2^K$, where K is a closed interval in $\mathbb{R}$. For any point to set mapping f and $\epsilon > 0$, we can find a continuous function that is within ϵ of the graph of f.

in the triangulation. Therefore, by taking finer and finer triangulations of S (with the mesh tending to zero), we obtain a sequence of completely labelled simplices (each simplex in this sequence is from one of these triangulations). Since S is compact, there exists a subsequence of the completely labelled simplices whose vertices all converge to some point $x \in S$. Using the upper semicontinuity f as well as the properties of the completely labelled simplices converging to x, we show that x has to be contained in $f(x)$.

We first define the labelling rule that we are going to use in the proof and show that it is a proper Sperner labelling: Let f be a convex and compact-valued point-to-set mapping from S to itself. Since $S \subset \mathbb{R}^k$ is both compact and convex, then by Lemma 3.8, for any $m \in \mathbb{N}$, there exists a continuous function $g^m : S \to S$ such that $Gr(g^m) \subset N_{1/m}(\mathrm{Gr}\,(f))$.

For each point $x \in S$, let $(x)_i$ denote the i^{th} barycentric coordinate of x. We may give a label i to x if v^i is a vertex of the carrier of x (that is, if $(x)_i > 0$) and $(x)_i \geq (g^m(x))_i$.

Proposition 3.9. *The labelling above is a proper Sperner labelling.*

Proof. Note that it is not possible that $(x)_i < (g^m(x))_i$ for all values of i. So, for each x, there exists some i such that $(x)_i \geq (g^m(x))_i$ and $(x)_i > 0$. $\square$

Then, we obtain the following result.

Theorem 3.10. *Sperner's lemma implies Kakutani's fixed point theorem.*

Proof. Given any triangulation of S, then we can label each vertex of the triangulation using the labelling defined above. That is, we fix a number $m \in \mathbb{N}$, and construct a continuous function g^m as specified above, that is contained in an ϵ-neighbourhood of the graph of f.

By proposition 3.9, we show that the labelling satisfies the hypothesis of Sperner's lemma. So, in any triangulation of S where each vertex is given a label in this way, there is a simplex that is completely-labelled. That is, its vertices receive $d+1$ distinct labels. Taking a sequence of finer and finer triangulations, we obtain a sequence of completely-labelled simplices whose mesh goes to zero. Call the simplices in the sequence Δ_n and denote the vertex labelled i as x_n^i. So, for each label i, we have a sequence $\{x_n^i\}$ of vertices. Since S is compact, for each of the $d+1$ such sequences (each is a sequence of vertices of the same label from the completely-labelled simplices), there is a subsequence that converges to a limit point, call it x_0^i. Since the mesh of Δ_n goes to zero, however, it is easy to see that in fact the limit points x_0^i of the sequences is the same for all i. So, we can refer to this limit point as simply x_0.

Now, we show that x_0 may be given any of the $d+1$ labels: Let i be an arbitrary label. We know that $x_n^i \to x_0$. Since g^m is continuous, then

$$g^m(x_n^i) \to g^m(x_0).$$

Since x_n^i receives the label i, then we know that

$$(x_n^i)_i \geq (g^m(x_n^i))_i,$$

for all n.

Suppose to the contrary that x_0 cannot be labelled i. Then,

$$(x_0)_i < (g^m(x_0))_i.$$

Let $d = (g^m(x_0))_i - (x_0)_i$. Choosing $0 < \epsilon < d/2$, then for all $x_n^i \in N_\epsilon(x_0)$ and $g^m(x_n^i) \in N_\epsilon(g^m(x_n^i))$, we have

$$(x_n^i)_i < (g^m(x_n^i))_i,$$

which is a contradiction. So, $(x_0)_i \geq (g^m(x_0))_i$. Since i is arbitrary, this is true for any of the $k+1$ labels, such that x_0 may receive any of them.

Since $(x_0)_i \geq (g^m(x_0))_i$ holds for all i, we know that in fact,

$$(x_0)_i = (g^m(x_0))_i.$$

This means that $x_0 = g^m(x_0)$. By Lemma 3.8,

$$(x_0, g^m(x_0)) \in N_{1/m}(\mathrm{Gr}\,(f)).$$

That is, each fixed point of g^m lies within $1/m$ of the graph of f.

Note that for different values of m, we obtain different functions g^m and different fixed points x_0 of g^m. So, we now rename the fixed point x_0 of g^m as $x_{0,m}$, for each $m \in \mathbb{N}$. Let us consider the sequence $\{x_{0,m}\}_{m \in \mathbb{N}}$. Since S is a compact space, then there exists a converging subsequence. Let $x_{0,0}$ denote the limit point of such subsequence. Note that

$$g^m(x_{0,m}) = x_{0,m}$$

for each m, so

$$(x_{0,m}, x_{0,m}) \in N_{1/m}(\mathrm{Gr}\,(f)).$$

Since Gr (f) is closed, then the limit point of $(x_{0,m}, x_{0,m})$, which is $(x_{0,0}, x_{0,0})$, is contained in the intersection:

$$\bigcap_{m \in \mathbb{N}} N_{1/m}(\mathrm{Gr}\,(f)).$$

Therefore, $(x_{0,0}, x_{0,0})$ is contained in Gr (f). So,

$$x_{0,0} \in f(x_{0,0}),$$

which implies that $x_{0,0}$ is a fixed point of f. □

3.4 Conclusion

The original proof of Kakutani's fixed point theorem by Shizuo Kakutani uses Brouwer's fixed point theorem. In this chapter, we have proven Kakutani's theorem using Sperner's lemma and an additional result namely Lemma 3.8. In a sense, this proof is directly from Sperner's lemma since we never invoke Brouwer's fixed point theorem. However, it is quite easy to notice that we have not completely avoided Brouwer's theorem.

To see this, note that we use Lemma 3.8 to obtain a sequence of continuous functions on the d-simplex S that are close to the given point-to-set mapping. We then use Sperner's lemma to show that there exists a fixed point for each function. Finally, we show that the limit point of the sequence of fixed points is a fixed point of the point-to-set mapping. Note that the fact that each function in the sequence has a fixed point is given by

Brouwer's fixed point theorem, since these functions are continuous. In the proof above, we do not invoke Brouwer's theorem, but in essence we use Sperner's lemma to re-prove Brouwer's theorem.

One of our motivations in trying to find a direct proof of Kakutani's theorem from Sperner's lemma is to obtain a constructive method to find a Kakutani fixed point via triangulations of the simplicial domain. Although we have a method to find the fixed point whose existence asserted in Kakutani's fixed point, our method is not entirely constructive unless we refer to the proof of Lemma 3.8, which suggests a way to construct the continuous function whose existence asserted by the lemma.

Chapter 4

Polytopal Sperner Lemma and von Neumann Intersection Lemma

In this chapter, we consider a result related to Kakutani's fixed point theorem which we shall refer to as von Neumann's intersection lemma. In [2], Kakutani's theorem is stated as a special case of von Neumann's lemma. In Kakutani's paper ([7]), however, von Neumann's lemma comes as a corollary to Kakutani's fixed point theorem. Since one can be proven using the other without much other tools besides basic analysis, then the two results are equivalent. Therefore, it is natural to also ask whether von Neumann's lemma can also be proven directly using Sperner's lemma.

Before going further, we state von Neumann's lemma as stated in [2].

Theorem 4.1 (von Neumann's Intersection Lemma)**.** *Let $K \subset \mathbb{R}^m$ and $L \subset \mathbb{R}^n$ be two bounded closed convex sets, and consider their Cartesian product $K \times L \in \mathbb{R}^{m+n}$. Let U and V be two closed subsets of $K \times L$ such that for any $x_0 \in K$, the set $U_{x_0} = \{y \in L | (x_0, y) \in U\}$ is nonempty, closed, and convex, and such that for any $y_0 \in L$, the set V_{y_0} of all $x \in K$ such that $(x, y_0) \in V$, is nonempty, closed, and convex. Under these assumptions, $U \cap V \neq \emptyset$.*

Since the case when K and L are subsets of $\mathbb{R}$ is much simpler than the general cases, we first present a proof of this case first in Section 4.1. Although we will not prove the general case of the lemma, we will present a possible approach to do so in Section 4.2. In both cases, we prove (respectively, give a proof idea) of the theorem not from the original Sperner's lemma, but from one of its generalizations. Specifically, we used the poly-

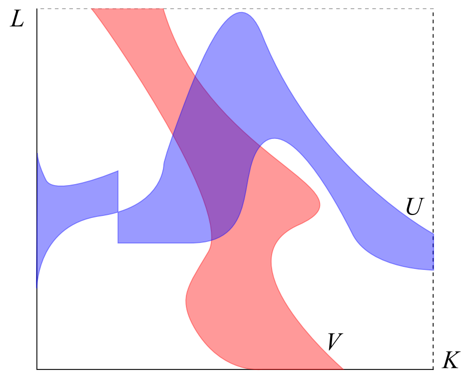

Figure 4.1: An illustration of Lemma 4.1 in $\mathbb{R} \times \mathbb{R}$

topal Sperner's lemma (Theorem 2.11) which we previously stated in Section 2.4.2.

4.1 The Case in $\mathbb{R} \times \mathbb{R}$

The case when K and L are both convex and closed subsets of $\mathbb{R}$ is the most simple case of lemma 4.1 and is the easiest to visualize. In the $\mathbb{R}$, any closed and convex subset is a closed interval. Hence, K and L of $\mathbb{R}$ are closed intervals on the real line, so their Cartesian product is a closed rectangle in $\mathbb{R}^2$. The following example illustrates the lemma for $K, L \subset \mathbb{R}$

Example 4.2. Let K and L each be a closed interval in $\mathbb{R}$. Consider the subsets U, V of $K \times L$ in Figure 4.1 and note that each "slice" U_{x_0} and V_{y_0}—the slice of U at x_0 and the slice of V at y_0 respectively—is a (nonempty) closed interval in $\mathbb{R}$, which is nonempty, closed, and convex for any x_0 in K and y_0 in L. Hence, U and V satisfies the hypothesis of Lemma 4.1. As concluded by the lemma, we observe that U and V have a nonempty intersection.

We now prove the $\mathbb{R} \times \mathbb{R}$ case of the lemma.

Proof. Let $K \subset \mathbb{R}, L \subset \mathbb{R}$ where K, L compact (closed and bounded intervals) and convex. $U, V \subset K \times L$ as described in Theorem 4.1.

Suppose $K = [a, b]$, $L = [c, d]$. Then, we can picture $K \times L$ as a closed rectangle. We will label each point in $K \times L$ with a 2-dimensional vector. In particular, we label the four vertices of this rectangle as follows: label the vertex (a,c) with $(0,0)$, (b,c) with $(1,0)$, (a,d) with $(0,1)$, and (b,d) with $(1,1)$.

Then, we label each point $(x,y) \in K \times L$ as follows. We assign 0 or 1 to each x and y and label (x,y) with the label of x followed by the label of y:

1. If $y \in U_x$, assign either 0 or 1 to y.

2. If $y \notin U(x)$ but there exists $y' \in U(x)$ such that $y' > y$, then label y with 0.

3. If $y \notin U(x)$ but there exists $y' \in U(x)$ such that $y' < y$, then label y with 1.
   (Note that since $U(x)$ is convex, exactly one of three cases above would be satisfied).

4. Similarly, if $y \in U_y$, assign either `0` or `1` to x.

5. If $x \notin U(y)$ but there exists $x' \in U(x)$ such that $x' > x$, then label x with 0.

6. If $x \notin U(y)$ but there exists $x' \in U(x)$ such that $x' < x$, then label x with 1.
   (Similarly, we note that since $U(y)$ is convex, exactly one of the three cases above would be satisfied.)

Visually, if x receives a label 0, then either x is in $U(y)$ or x is 'below' $U(y)$. Similarly, if y receives a label 0, then either y is in $V(y)$ or y is "to the left" of $V(y)$.

This is a Sperner labelling because each point in a triangulation can receive a label and if (x,y) lies on a proper face (a side) of $K \times L$, then it can only receive one of the two labels of the vertices that spans that side.

If we triangulate $K \times L$ and label the vertices of this triangulation with the labelling above, then by [6][Theorem 1], there exists a fully labelled simplex. That is, there is a 2-simplex with three (of the four) distinct labels. Taking a sequence of finer and finer triangulations of $K \times L$, we obtain a sequence of fully labelled triangles. Since this sequence contains infinitely many triangles and only finitely many different possible combinations of set of three labels, then there is at least one set of three distinct labels that appears infinitely many times in the sequence. Take this subsequence of triangles. Since the space is compact, there is a converging subsequence.

Suppose that $\{\Delta_n\}$ denote the converging subsequence of fully labelled triangles and Δ_n converges to the point (x_0, y_0). Without loss of generality, let the label set of this sequence be $\{(0,0),(0,1),(1,0)\}$. We wish to show that (x_0, y_0) can receive any of these three labels.

We first note that by the criteria of labelling we specify above, the set of points that can receive a given label is closed.

Consider the vertex of each Δ_n the receives the label $(0,0)$, let's say. Since (x_0, y_0) is the limit point of a sequence of points that are labelled $(0,0)$ and since the set of points that can receive this label is closed, then (x_0, y_0) can also be labelled with $(0,0)$.

Since (x_0, y_0) can be labelled with both $(0,0)$ and $(0,1)$, then (x_0, y_0) must lie in U. Similarly, since (x_0, y_0) can be labelled with both $(0,0)$ and $(1,0)$, then (x_0, y_0) must lie in V. Hence, (x_0, y_0) lies in the intersection of U and V. $\square$

4.2 The General Case

In this section, we present a possible approach to prove the general case of Lemma 4.1 using the polytopal Sperner lemma, for subsets $K \subset \mathbb{R}^m$ and $L \subset \mathbb{R}^n$ which are m and n-dimensional simplices, respectively. Note that the Cartesian product $K \times L$ is an $(m+n)$-dimensional polytope. We consider subsets U and V of $K \times L$ that satisfies the hypothesis of the lemma.

The basic idea is similar to the proof of the specific case in the previous section. Namely, we first triangulate the polytope $K \times L$. Then, we label each vertex (x, y) in the triangulation of $K \times L$ with a 2-dimensional vector (l_1, l_2), where l_1 is the label of $x \in K$ and l_2 is the label of $y \in L$. Note that the labelling of each point $x \in K$ will be given with respect to $U(x)$, while the labelling of each point $y \in L$ will be given with respect to $V(y)$. We will show that if the labelling in each of the factor space is a proper Sperner labelling, then the vector labelling in the product space is also a proper Sperner labelling.

Then, using polytopal Sperner lemma, we know that there exist at least $(mn - m - n)$ fully-labelled simplices (since the polytope $K \times L$ has mn vertices and is $(m+n)$-dimensional), whose image under the map f (defined in Proposition 2.12) forms a cover of $K \times L$. Taking a sequence of triangulations with mesh converging to zero, we obtain a converging subsequence of fully-labelled simplices of a given set of vertex labels. We use the labelling properties in each of the factor space, to show that the limit point of the subsequence must lie in the intersection of U and V.

Following the proof outline above, we first specify a method for labelling a point in each of the factor spaces. We should be able to use a labelling method that is essentially the same labelling method that we use in Section 3.3. That is, for each $m \in \mathbb{N}$, we use a continuous function $g^m : K \to L$ whose graph lies within $1/m$ of U. Similarly, we use a continuous function $h^m : L \to K$ whose graph lies within $1/m$ of V.

In order to guarantee the existence of such continuous functions $\{g^m\}, \{h^m\}$, however, we need to show that the following point-to-set mappings satisfies the hypothesis of Lemma 3.8:

$$u : K \to 2^L,$$

$$v : L \to 2^K,$$

given by

$$u(x) = \{y \in L | (x,y) \in U\},$$
$$v(y) = \{x \in K | (x,y) \in V\}.$$

Note that $u : K \to 2^L$ and $v : L \to 2^K$ are in fact slices of U and V. Once we show that u and v satisfies the hypothesis of Lemma 3.8, namely upper semicontinuous, with nonempty, convex, and compact values, and obtain continuous functions $\{g^m\}$ and $\{h^m\}$, we use a labelling function similar those in the proof of Kakutani's theorem from Sperner's lemma in Section 3.3 in each factor space.

Finally, on a triangulation of $K \times L$, we label each vertex of the triangulation using a 2-dimensional vector, whose components are the labels of the points in each factor space.

After obtaining a sequence of finer and finer triangulations and hence a sequence of fully-labelled simplices that converges to a point, we then show that the limit point can be given any of the possible mn labels. Finally, we hope to use this to show that it must be contained in the intersection of U and V.

4.3 Conclusion

In this chapter, we present a proof of a special case of the intersection lemma in $\mathbb{R} \times \mathbb{R}$. Although we have not been able to prove the general case, we give a possible method for proving the general case using the polytopal Sperner's lemma together with the approximation lemma that we also used in Chapter 3.

Chapter 5

Discussion and Future Work

In this paper, we present our main result, a proof of Kakutani's fixed point theorem directly from Sperner's lemma. However, there are still some questions that are left open and some possible improvements that could be done. We discuss our results and describe future work that could continue from this thesis.

5.1 Discussion

Proving Kakutani's theorem directly from Sperner's lemma prove to be more challenging than we thought. In Chapter 3, we present preliminary work which include a proof of a weaker version of Kakutani's theorem from Sperner's lemma (Theorem 3.2). The proof of this result utilizes a method that was not sufficient in proving the original Kakutani's theorem but is enough to prove a weaker version of Kakutani's theorem. In particular, we impose the additional "continuity" condition that is stronger than upper semicontinuity required in Kakutani's theorem.

In Section 3.2, we also presented one additional labelling method that has failed to work due to a similar problem. Namely, the labelling properties are not strong enough to ensure that any limit point of a converging sequence of completely-labelled simplices to be a fixed point of the mapping.

Finally, in Section 3.3, we are able to find a labelling function that works and thus complete the proof of Kakutani's theorem from Sperner's lemma.

5.2 Future Work

We are able to complete a proof of Kakutani's theorem directly from Sperner's lemma. However, as we discuss at the end of Chapter 3, this proof has not completely avoided the use of Brouwer's fixed point theorem and requires finding a continuous function near the graph of the point-to-set mapping. A modification of the current proof that allows us to completely bypass the use of a continuous function and Brouwer's fixed point theorem would be a desirable goal for a future work on this topic.

In Chapter 4, we provide a proof for the simple case in $\mathbb{R} \times \mathbb{R}$ of the intersection lemma (Theorem 4.1) and an idea for a proof of the general case. Completing the proof of the intersection lemma using the polytopal Sperner's lemma would be another possible direction for future work.

Appendix A

Proof of Approximation Lemma

In this appendix, we provide the proof of Lemma 3.8, taken from [2]. We first recall the lemma.

Lemma (von Neumann's Approximation Lemma, 1937)**.** *Let $\gamma : E \to 2^F$ be an upper semicontinuous point-to-set mapping with nonempty compact, convex values, where $E \subset \mathbb{R}^m$ is compact and $F \subset \mathbb{R}^k$ is convex. Then for any $\epsilon > 0$ there is a continuous function f such that $Gr(f) \subset N_\epsilon(Gr(\gamma))$.*

The proof of the above lemma uses the following result.

Lemma A.1 (Cellina, 1969)**.** *Let $\gamma : E \to 2^F$ be an upper semicontinuous point-to-set mapping with nonempty compact, convex values, where $E \subset \mathbb{R}^m$ is compact and $F \subset \mathbb{R}^k$ is convex. For $\delta > 0$, define γ^δ by:*

$$\gamma^\delta(x) = co \bigcup_{z \in N_\delta(x)} \gamma(z).$$

Then, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$Gr\, \gamma^\delta \subset N_\epsilon(Gr\, \gamma).$$

Of Lemma A.1. Suppose to the contrary, then for each $n \in \mathbb{N}$, there must exists $(x^n, y^n) \in \mathrm{Gr}\, \gamma^{1/n}$ such that $d((x^n, y^n), \mathrm{Gr}\, \gamma) \geq \epsilon > 0$. Note that $(x^n, y^n) \in \mathrm{Gr}\, \gamma^{1/n}$ means

$$y^n \in \gamma^{1/n}(x^n)$$

which implies that

$$y^n \in \mathrm{co} \bigcup_{z \in N_{1/n}(x^n)} \gamma(z).$$

By Caratheodory's theorem, there exist $y^{0,n}, \ldots, y^{k,n} \in \bigcup_{z \in N_{1/n}(x^n)} \gamma(z)$ such that

$$y^n = \sum_{i=0}^{k} \gamma_i^n y_i^{i,n}$$

with $\gamma^i \geq 0$, $\sum_{i=0}^{k} \gamma^i = 1$, and $y^{i,n} \in \gamma(z^{i,n})$ with $|z^{i,n} - x^n| < \frac{1}{n}$. Since E is compact and γ is upper semicontinuous, we can extract convergent sequences such that $x^n \to x$, $y^{i,n} \to y^i$, $\gamma_i^n \to \gamma_i$, $z^{i,n} \to x$ for all i, and $y = \sum_{i=0}^{k} \gamma_i y^i$ and $(x, y^i) \in \mathrm{Gr}\,\gamma$ for all i. Since γ is convex-valued, $(x, y) \in \mathrm{Gr}\,\gamma$, which contradict $d((x^n, y^n), \mathrm{Gr}\,\gamma) \geq \epsilon$ for all n. □

Finally, we can now prove Lemma 3.8.

Proof. By Lemma A.1, there is a $\delta > 0$ such that the correspondence γ^δ satisfies $\mathrm{Gr}\,(\gamma^\delta) \subset N_\epsilon(\gamma)$. Since E is compact, there exists $x^1, \ldots, x^n$ such that $\{N_\delta(x^i)\}$ is an open cover of E. Choose $y^i \in \gamma(x^i)$. Let $f^i, \ldots, f^n$ be a partition of unity subordinate to this cover and set $g(x) = \sum_{i=0}^{n} f^i(x) y^i$.

Then g is continuous and since f^i vanishes outside $N_\delta(x^i)$, then $f^i(x) > 0$ implies that $|x^i - x| > \delta$ so $g(x) \in \gamma^\delta(x)$. □

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