Abelian Sandpile Model: Symmetric Sandpiles

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November 16, 2008
Self Organized Criticality

In an equilibrium system the critical point is reached by tuning a control parameter precisely. Example: Melting water.

**Definition**

**Self-Organized Criticality** A *phenomenon of a non-equilibrium system with a critical point as attractor. Behavior of critical points same as phase transition, but without control parameters.*
Self Organized Criticality

Sand piles exhibit this behavior: static periods with intermittent sand slides. Catastrophes are inevitable, the product of minor events in the past.

Figure: From *How Nature Works*, Per Bak.
Self Organized Criticality

- Punctuated Equilibrium (Sand avalanches correspond to cladogenesis, rapid events of branching speciation)
- Modeling Evolution (No need for meteors to explain dinosaur extinction)
- The Brain (Thought is a critical state, an avalanche triggered by visual stimulus or another thought)
- Economics (Network of consumers and producers, supply and demand avalanches. Critical economy vs. equilibrium economy)
Directed graph, $\Gamma = (V, E, s)$ (multiple edges and self loops OK).

- $d_{v_1} = \text{outdegree}(v_1) = 2$
- $d_s = 0$
- $s$ is a global sink
The Model

Directed graph, $\Gamma = (V, E, s)$ (multiple edges and self loops OK).

- Laplacian

$$\Delta = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

- Reduced Laplacian

$$\Delta' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
Sandpiles

Sandpile: Integer weighting on the non-sink vertices of $\Gamma$

- $\sigma \in \mathbb{N}^2$
- $\sigma = (2, 1)$
- $\sigma(v_1) = 2$, $\sigma(v_2) = 1$
- $\sigma$ unstable at $v$ if $\sigma(v) \geq d_v$
- $\sigma$ is unstable at $v_1$
Firing unstable vertices: Recall the Reduced Laplacian

$$\Delta' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\sigma = (2, 1)$$
Firing unstable vertices: Recall the Reduced Laplacian

\[ \Delta' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]

\[ \sigma = \begin{pmatrix} 2, 1 \\ 2, -1 \end{pmatrix} - \begin{pmatrix} 0, 2 \end{pmatrix} \]
Stabilization

Firing unstable vertices: Recall the Reduced Laplacian

\[
\Delta' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
\]

\[
\sigma = \begin{pmatrix} (2, 1) \\ (2, -1) \end{pmatrix}
\]

\[
(0, 2)
\]
Stabilization

Firing unstable vertices: Recall the Reduced Laplacian

$$\Delta' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} (2, 1) \\ (2, -1) \end{pmatrix} - \begin{pmatrix} (0, 2) \\ (-1, 2) \end{pmatrix}$$

$$\sigma^\circ = \begin{pmatrix} (1, 0) \end{pmatrix}$$
Firing unstable vertices: Recall the Reduced Laplacian

\[ \Delta' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]

\[ \sigma = (2, 1) \]
\[ - (2, -1) \]
\[ (0, 2) \]
\[ - (-1, 2) \]

\[ \sigma^\circ = (1, 0) \]

\(\sigma^\circ\) is the stabilization of \(\sigma\).
Theorem

If $\Gamma$ has a global sink, every configuration on $\Gamma$ stabilizes.
Sand Addition Operator

**Definition**

For \( v \in V \setminus s \), the sand addition operator \( E_v \) acts on configurations as follows:

\[
E_v \sigma = (\sigma + 1_v)^\circ
\]

For digraph with global sink, sand addition operator commutes, i.e.

\[
E_w E_v \sigma = (\sigma + 1_w + 1_v)^\circ = (\sigma + 1_v + 1_w)^\circ = E_v E_w \sigma
\]

Applying a sequence of sand additions to \( \sigma \) yields same result as adding all sand simultaneously and then stabilizing.
The Sandpile Group

• Let $\Delta'$ denote the reduced Laplacian of $\Gamma$, with $n$ the number of non-sink vertices. The Sandpile group of $\Gamma$ is given by:

$$S(\Gamma) = \mathbb{Z}^{n-1}/\mathbb{Z}^{n-1}\Delta'(\Gamma)$$

• $S(\Gamma)$ is the integer row span of the reduced Laplacian.
• $|S(\Gamma)| = \det(\Delta')$
Recurrent Configurations

Definition

A configuration $\sigma$ is accessible if for all configurations $\alpha$, there exists a $\beta$ such that $\alpha + \beta \rightarrow \sigma$. If $\sigma$ is stable and accessible, then $\sigma$ is recurrent.

• Every equivalence class of $S(\Gamma)$ contains a unique recurrent configuration.
• Set of all recurrent configurations on $\Gamma$ forms an abelian group under $(\sigma, \sigma') \mapsto (\sigma + \sigma')^\circ$ and is isomorphic to $S(\Gamma)$. 
The Identity Sandpile

\[ I = (\sigma - \sigma^\circ) \]

Figure: Identities for the 2 × 2 grid, 3 × 3 grid, and 5 × 5. Colors: red = 1, blue = 2, yellow = 3, pink = 4.
The Identity Sandpile: 57 by 57 Grid
The Six Grid

Γ₆ =

\[ \tilde{\Delta} = \begin{pmatrix}
4 & -1 & 0 & -1 & 0 & 0 & -1 \\
-1 & 5 & -1 & 0 & -1 & 0 & -1 \\
0 & -1 & 4 & 0 & 0 & -1 & -1 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 \\
0 & -1 & 0 & -1 & 5 & -1 & -1 \\
0 & 0 & -1 & 0 & -1 & 4 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 6
\end{pmatrix} \]
Group Action Partitions Vertices into Equivalence Classes

\[ G \text{ on } \Gamma_6 = \]

\[ \tilde{\Delta} = \begin{pmatrix}
4 & -1 & 0 & -1 & 0 & 0 & -1 \\
-1 & 5 & -1 & 0 & -1 & 0 & -1 \\
0 & -1 & 4 & 0 & 0 & -1 & -1 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 \\
0 & -1 & 0 & -1 & 5 & -1 & -1 \\
0 & 0 & -1 & 0 & -1 & 4 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 6
\end{pmatrix} \]
### Obtaining the Quotient Graph

<table>
<thead>
<tr>
<th>Equivalence Class</th>
<th>$(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${v_1, v_3, v_4, v_6}$</td>
<td>$(3, -2, 3, 3, -2, 3, -4)$</td>
</tr>
<tr>
<td>${v_2, v_5}$</td>
<td>$(-1, 4, -1, -1, 4, -1, -2)$</td>
</tr>
<tr>
<td>${v_7}$</td>
<td>$(-1, -1, -1, -1, -1, -1, 6)$</td>
</tr>
</tbody>
</table>

\[ \Delta_q = \begin{pmatrix} 3 & -2 & -4 \\ -1 & 4 & -2 \\ -1 & -1 & 6 \end{pmatrix} \quad \Delta_T = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 4 & -1 \\ -4 & -2 & 6 \end{pmatrix} \]

Let $\Gamma_q$ denote the graph described by $\Delta_T$. 
Symmetric Elements

Definition

\( \sigma \in S(\Gamma_6) \) is symmetric if it is of the form:

\[
(\sigma(v_1), \sigma(v_2), \sigma(v_1), \sigma(v_1), \sigma(v_2), \sigma(v_1), \sigma(v_7))
\]

- The symmetric elements of \( S(\Gamma_6) \) form a subgroup, \( S(\Gamma_6)^G \).
- We also calculate the recurrent configurations that comprise \( S(\Gamma_q) \) for the quotient graph.
Symmetric Subgroup and Laplacian of the Quotient Graph

- The sandpile group $S(\Gamma_q)$ has the same number of elements as the symmetric subgroup of the original graph!
- $|S(\Gamma_q)|$ is given by the determinant of $\Delta_q^T$. 
Problems with the Quotient Graph

- The elements of $S(\Gamma_q)$ do not match the elements of $S(\Gamma_6)^G$.
- Why did we need to use the transpose?
- Rows of $\Delta_q$ do not sum to zero.
- $\Delta_q$ should treat the entire equivalence class as a vertex.
Rescaling the Laplacian

\[ \Delta_q = \begin{pmatrix} 3 & -2 & -4 \\ -1 & 4 & -2 \\ -1 & -1 & 6 \end{pmatrix} \text{ reweighted } \rightarrow \begin{pmatrix} 12 & -4 & -4 \\ -4 & 10 & -2 \\ -4 & -2 & 6 \end{pmatrix} \]

- Note that \( \Delta'_q \) is symmetric like a Laplacian should be.
- \( |S(\Gamma'_q)| = 352 \) as opposed to 30. Too many elements!
- Configurations can have up to 11 sand grains on a vertex and be stable.
...And Normalizing?

\[ \Delta'_q = \begin{pmatrix} 12 & -4 & -4 \\ -4 & 10 & -2 \\ -4 & -2 & 6 \end{pmatrix} \rightarrow \parallel \Delta'_q \parallel /4 = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 5/2 & -1/2 \\ -1 & -1/2 & 3/2 \end{pmatrix} \]

\[ \rightarrow \parallel \Delta'_q \parallel /2 = \begin{pmatrix} 6 & -2 & -2 \\ -2 & 5 & -1 \\ -2 & -1 & 3 \end{pmatrix} \]

- The graph corresponding to \( \parallel \Delta'_q \parallel /4 \) has only 11 elements.
- The graph corresponding to \( \parallel \Delta'_q \parallel /2 \) has 44 elements.
- Straying too far from our original construction.
Future Exploration

- Does there exist an isomorphism between $S(\Gamma)^G$ and $S(\Gamma_q)$?
- More generalized notion of firing that allows us to dip negative
Future Exploration

- Does there exist an isomorphism between $S'(\Gamma)^G$ and $S'(\Gamma_q)$?
- More generalized notion of firing that allows us to dip negative
Any Questions?

- Professor Su (Advising)
- Professor Perkinson (Advising)