Continued Fractions: A New Form

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Introduction

Continued fractions are a way to approximate real numbers as tuples of integers. For instance, Figure 1 shows the most common form of continued fractions, in which each successive \( a_n \) except the first, is a positive integer (Khinchin, 1992).

\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}}
\]

**Figure 1:** Ordinary continued fractions

In certain scenarios, however, a different approximation might be more useful. One such approximation is the basis for this thesis. The new approximation takes the form described in Figure 2. Here, each successive \( t_i \) is a positive integer with \( t_i \geq 2 \), and \( t_i > t_{i-1} \).

\[
1 + \frac{(-1)^{t_i}}{1 + \frac{1}{t_i + \frac{1}{t_{i-1} + \ddots}}}
\]

**Figure 2:** The new continued fractions

These continued fractions approximate real numbers between 1 and 2. A algorithm for generating these fractions was described by Pippenger (Pippenger, 1979).

Materials and Methods

- The expression in Figure 3 is used to represent the expression in Figure 2.
- The number \( n \) in Figure 2 and Figure 3 is defined to be the order of the continued fraction.
- Some largely unsuccessful use of Wolfram Mathematica came near the end of the thesis.

\[
(t_1, t_2, \ldots, t_n)
\]

**Figure 3:** A more concise way of representing the new fractions

Results

From a tuple to a rational

Usually, analyses of continued fractions focus on the process of starting with a real number and finding the relevant continued fraction approximation. However, it is also useful to move the other way; that is, if we take a tuple of strictly increasing positive integers, we wish to know what rational number is represented by the corresponding continued fraction. The rational number generated by the integers \( t_1, t_2, \ldots, t_n \) is given in Figure 4.

\[
\frac{\prod_{k=1}^{n} a_k}{\sum_{i=0}^{n-1} (-1)^{i+n} \prod_{k=0}^{i} a_k}
\]

**Figure 4:** The rational number generated by a given tuple

This form proves more useful in analyzing the average- and worst-case errors of approximation.

Worst-case approximation

Define the error of order \( n \) for a given real number \( x \) to be the difference between \( x \) and the continued fraction of order \( n \) generated by the algorithm described by Pippenger. Then, we can define the worst-case approximation of order \( n \) to be the real number for which the error of order \( n \) is maximized.

The worst-case approximation of order \( n \) was found to be the number that bisects the interval between \((2, 3, \ldots, n, n + 1)\) and \((2, 3, \ldots, n, n + 2)\). As \( n \to \infty \), these bounds converge to \( \frac{\sqrt{5} - 1}{2} \).

Average-case approximation

If we take the reals from 1 to 2 from a uniform distribution, we can define the average-case approximation of order \( n \) to be the expected value of the error of order \( n \) of a random real number. When \( n = 1 \), the average error is equal to \( \frac{\sqrt{5} - 1}{2} \); see Figure 5.

\[
1 \sum_{i=2}^{n} \left( \prod_{k=2}^{i} a_k \right) = \frac{n^2 - 9}{12}
\]

**Figure 5:** The average error of order 1

When \( n = 2 \), the average error is more difficult to calculate. It is equal to the value of the summation shown in Figure 6; however, I have been unable to find a closed form.

\[
1 \sum_{i=2}^{n} \left( \prod_{k=2}^{i} a_k \right) \left( \prod_{k=2}^{i} a_k \right) = \frac{n^2 - 9}{12}
\]

**Figure 6:** The average error of order 2

Asymptotic behavior

In the absence of a general formula for the average-case error of a given order \( n \), it became necessary to examine the asymptotic behavior of the average-case error; that is, I looked for the behavior of the average-case error as \( n \) becomes arbitrarily large.

I found an upper bound on the complexity of the approximation; the average-case error of the new approximation is \( O(3^{-n}(n!)^{-3}) \).

Note that this is not a strict upper bound.

Conclusions

This thesis has described a number of properties of a newer form of continued fractions; this, hopefully, will allow the new form to be used more often more successfully. It should be noted that there are a number of important properties that are still unknown; looking only at those that were hinted at by this thesis, we see that there is no general formula for the average-case approximation of order \( n \), nor is there any kind of lower bound on the asymptotic behavior of the average-case error. Future work would attempt to answer these questions.

References


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- The full report and a PDF of this poster can also be found at the above website.