

**Math 197: Senior Thesis**

**Approval Ratios of Double-Interval Societies**

Jacob Scott

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**Introduction**

Helly’s theorem states that given any \( n \) convex sets in \( \mathbb{R}^d (d < n) \), if every \( d + 1 \) of them have nonempty intersections, then all \( n \) sets must have nonempty intersection:

\[ \text{Theorem 1. (Helly’s Theorem, 1913)} \]

If \( \{S_i\} \) is a collection of convex sets in \( \mathbb{R}^d \), \( 1 \leq i \leq n \), and the intersection of every \( d + 1 \) of the sets in \( \{S_i\} \) is nonempty, then \( \bigcap_{i=1}^{d+1} S_i \neq \emptyset \).

If each convex set represents an interval of political positions on a spectrum that a voter approves of, then Helly’s theorem establishes a necessary and sufficient condition to guarantee that some political position has an approval ratio of 1.

- Define a society to be a political spectrum together with a collection of voters’ approval sets; the size of a society is the number of such sets.
- Call a society \( 2 \)-intersecting if all its approval sets pairwise intersect.
- Define the **agreement number** and **approval ratio** of a society to be the maximum number and proportion, respectively, of mutually intersecting approval sets.

An example society is shown in Figure 1. Sets are shown separated vertically for visibility.

**Previous Work**

- Berg et al. (2010): Suppose approval sets are intervals in \( \mathbb{R}^1 \) and out of every collection of \( m \) approval sets, at least \( k \) mutually intersect. Then the approval ratio is at least \( \frac{k}{m} \).
- Hardin (2010): Now suppose approval sets are now arcs on a circle, and again that out of every collection of \( m \) approval sets, at least \( k \) mutually intersect. Then the approval ratio is at least \( \frac{k}{m} \).

What if approval sets were not just single intervals?

**Main Question**

- Call a society \( S \) double-\( i \)-interval if each approval set in \( S \) is the union of two closed intervals in \( \mathbb{R}^1 \).

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**Double-n Strings**

K. Nyman and F. Su conjectured that approval ratios of double-\( i \)-intersecting societies are always \( \geq \frac{1}{3} \) and are achieved when the intervals of the approval sets are equally spaced and of the same length, as in Figure 2. This suggests the following definitions.

- Define a double-\( n \) string as a string of length \( 2n \) that contains two occurrences of \( n \) distinct symbols.
- Let the **double-\( n \) string** be the number of such sets.
- Let \( \delta(n) \) be the maximum distance between symbols in a double-\( n \) string.

For example, the double-5 string ABCDEBECAD has approval number 3:

\[ \text{Figure 1: A double-\( i \)-intersecting society of size 4 with approval number 3:} B, C', and D mutually intersect, for example. Each voter has 2 approval sets, labeled by unprimed/primed versions, unprimed on the left. \]

**Double-n Strings**

\[ \text{Double-n Strings} \]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A' )</td>
<td>( B' )</td>
<td>( C' )</td>
<td>( D' )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A double-\( n \) string of diameter \( d \) corresponds to a double-\( i \)-intersecting society \( S \) of approval ratio \( \frac{1}{2} \).

**Theorem 2. (Klawe, 2008)**

1. For any double-\( n \) string \( w, n \leq 3d(w) - 1 \).
2. Asymptotically, \( \frac{24}{69} \leq \frac{\delta(n)}{n} \leq \frac{5}{13} \).

Unfortunately, one of our main findings is that double-\( n \) strings do not always have the lowest approval ratios. While Klawe’s results show no double-8 string could have approval number less than 4, we demonstrate in Figure 3 an example society of size 8 with approval number 3.

\[ \text{Figure 2: A society of approval number 3 corresponding to the double-5 string ABCDEBECAD.} \]

\[ \text{Figure 3: A society of size 8 with approval number 3.} \]

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**Combinatorial Approach**

For an interval \( X \) of an approval set in society \( S \) of size \( n \), define \( L(X), R(X), C(X), \) and \( B(X) \) to be the number of other intervals intersecting \( X \) that contain just \( X \)’s left endpoint, just \( X \)’s right endpoint, neither of \( X \)’s endpoints, and both of \( X \)’s endpoints respectively. For example, in Figure 1, \( L(A') = 1, R(A') = 2, C(A') = 0, \) and \( B(A') = 1 \).

- If the approval number of \( S \) is \( m \), \( L(A) + B(A) \) is bounded by \( M - 1 \).

\[ \text{Theorem 3. (Scott, 2010)} \]

1. Asymptotically, \( \frac{M}{n} \geq 2 - \sqrt{3} \approx 0.268 \).

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**Further Study**

There remains a significant gap between the best theoretical lower and upper bounds on approval ratios. Theorem 3 provides the first lower bound of the asymptotic approval ratio, which falls short of the hypothesized asymptotic ratio of \( \frac{1}{2} \).

We wrote a program that repeatedly permutes interval endpoints; the data suggest that, with high probability, the approval ratio is asymptotically \( \frac{1}{2} \). More work is needed to tighten the lower bound on approval ratios of double-\( i \)-intersecting societies.

**Acknowledgments**

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**References**

- Klawe, M., “Double Distance”, Notes from Presentation at Oakland University, MI.

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Advisor: Francis Su  
Reader: Nicholas Pippenger  
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