Riemann–Roch

The Riemann–Roch Theorem is an important result in Complex Analysis and Algebraic Geometry. Without going into any detail at all, it states that there exists a canonical divisor $K$ of degree $2g − 2$ such that $r(D) − r(K − D) = \deg(D) + g − 1$ for all divisors $D$. Asadi and Backman (2011) demonstrate the same result for strongly connected directed multigraphs, except with a different rank function, $\tau$. We can play the following chip-firing game on your graph:

- You start out assigning each vertex an integer (possibly negative) number of chips
- Vertices can fire, sending a chip along each of their outgoing edges
- Vertices can borrow, which is just the inverse of firing

This is demonstrated to the left. Starting with an initial configuration, we fire $v_2$, followed by $v_3$. Notice that this brings all vertices out of debt. This could also have been achieved by having $v_2$ borrow once.

Notice that we can (and will) associate configurations in this game to divisors of the graph, which will allow us to use the chip-firing game to study Riemann–Roch on graphs!

Row Chip–Firing Game

Think about your favorite strongly connected directed multigraph. (Mine is shown in the figure to your left.) We can play the following chip firing game on your graph:

- You start out assigning each vertex an integer (possibly negative) number of chips
- Vertices can fire, sending a chip along each of their outgoing edges
- Vertices can borrow, which is just the inverse of firing

This is demonstrated to the left. Starting with an initial configuration, we fire $v_2$, followed by $v_3$. Notice that this brings all vertices out of debt. This could also have been achieved by having $v_2$ borrow once.

Notice that we can (and will) associate configurations in this game to divisors of the graph, which will allow us to use the chip-firing game to study Riemann–Roch on graphs!

Then the degree of a divisor (configuration) is just the total number of chips on the graph.

We say that two divisors are equivalent if you can get from one to the other with some sequence of firings and borrowings, or equivalently if they differ by a point in the lattice spanned by the rows of the directed Laplacian matrix $Q$.

A natural question to ask is, given an initial configuration $D$, can we bring all of the vertices out of debt? Put another way, is $D$ equivalent to a non-negative divisor (which we call effective)?

To this end, we now define:

Definition. The rank of a divisor $D$, denoted $r(D)$, is $−1$ if there is no firing strategy that brings all vertices out of debt. Otherwise, $r(D)$ is the largest non-negative integer $r$ such that any way of removing of $r$ chips from the game still results in a configuration that can be brought out of debt by some firing strategy.

Extreme Divisors

We now introduce something with proves to be very important. We have been able to achieve a canonical divisor $K$ of degree $2g − 2$ such that $r(D) − r(K − D) = \deg(D) + g − 1$ for all divisors $D$. Asadi and Backman (2011) demonstrate the same result for strongly connected directed multigraphs, except with a different rank function, $\tau$. We can play the following chip-firing game on your graph:

- You start out assigning each vertex an integer (possibly negative) number of chips
- Vertices can fire, sending a chip along each of their outgoing edges
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This is demonstrated to the left. Starting with an initial configuration, we fire $v_2$, followed by $v_3$. Notice that this brings all vertices out of debt. This could also have been achieved by having $v_2$ borrow once.

Notice that we can (and will) associate configurations in this game to divisors of the graph, which will allow us to use the chip-firing game to study Riemann–Roch on graphs!

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Definition. The rank of a divisor $D$, denoted $r(D)$, is $−1$ if there is no firing strategy that brings all vertices out of debt. Otherwise, $r(D)$ is the largest non-negative integer $r$ such that any way of removing of $r$ chips from the game still results in a configuration that can be brought out of debt by some firing strategy.

Definition. An extreme divisor is one that is not equivalent to an effective divisor, but which is if even a single chip is added to any vertex.

We can look at these divisors geometrically through the following observation: A divisor is equivalent to an effective if and only if it can be formed from adding chips to some point in the lattice spanned by the rows of the Laplacian.

$v^*$-Reduced Divisors

For this reason, it would be incredibly useful to get our hands on the extreme divisors of a graph, at least up to equivalence, should we want to determine if it has the Riemann–Roch property or not. To do this, we define:

Definition. A divisor $D$ of a graph $G$ is $v^*$-reduced if:

(i) for all $v \in V(G) \setminus v^*, D(v) \geq 0$.
(ii) carrying out any valid firing strategy will bring some vertex other than $v^*$ into debt.

Lemma (Lemma 3.10 of Asadi and Backman (2011)). Let $G$ be a directed graph and let $D$ be a divisor. Then

(i) $D$ is equivalent to an effective divisor if and only if there exists a $v^*$-reduced divisor $D' \sim D$ such that $D'$ is effective;
(ii) Suppose $D$ is not equivalent to an effective divisor. Then $D$ is an extreme divisor if and only if for any $v \in V(G)$, there exists a $v^*$-reduced divisor $D' \sim D$ such that $D'(v) = −1$.

This allows us to restrict the search to testing finitely many candidates.

References


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For Further Information

• Email me at Zachary_Gaslowitz@hmc.edu
• Check out the full thesis at http://www.math.hmc.edu/~jgaslowitz/thesis/