Introduction

Definition 1. A cap set of width \( n \) is a subset \( S \subset Z^n \) such that if \( x, y, z \in S \) sum to 0, then \( x = y = z \). Let \( a_n \) denote the size of the largest cap set of width \( n \).

It is natural to ask how quickly \( a_n \) grows. Another reason for interest in its growth is that it is related to the existence of efficient algorithms for matrix multiplication [CKSU05, ASU12]. The best known upper and lower bounds are respectively \( O(3^n/\sqrt{n}) \) and \( 2.2174^n \). The upper bound is due to Meshulam [Mes95, Rot53]. [Ede04] proves the lower bound.

A New Approach

In this section, we describe a new approach to bound the union of large subspaces of \( Z^n \). The union of large subspaces of \( Z^n \) should be viewed as a feature that is different from the exponential function describing the size of the best known construction of a related combinatorial object called a strongly resolution of a solitaire puzzle [CKSU05].

Conjecture 1. Let \( S \subset Z^n \) where \( S \subset \omega(3^n/n) \) and \( S \) is the union of \( S \)\!/\!n disjoint bases for \( Z^n \).

If \( S \) is conjectured to be true, we immediately obtain an upper bound on the size of the cap set. The conjecture is by way of the stronger one.

Conjecture 2. Let \( S \subset Z^n \) where \( S \subset \omega(3^n/n) \) and \( S \) is the union of \( S \)\!/\!n disjoint bases for \( Z^n \).

Then, \( |S| = 3^n - |S| \).

Conjecture 3. Let \( S \subset Z^n \) where \( S \subset \omega(3^n/n) \) and \( S \) contains a basis for \( Z^n \).

Then, \( |S| > 3^n - |S| \).

It may be worth investigating what other features of cap sets—aside from the fact that they have a lot of disjoint bases—might be useful in obtaining bounds.

Some Additional Conjectures

[ASU12] showed that constructing large sunflower free sets implies the existence of large cap sets. Briefly, Definition 3. A sunflower free set of width \( n \) is a subset \( S \) of \( F^n \) such that for any \( x, y, z \in S \) (not all equal), there is an \( i \) such that exactly two of \( x_i, y_i, z_i \) are equal to 1. Let \( b_n \) denote the size of the largest sunflower free set of width \( n \).

Theorem 2. [ASU12]. If \( \lim_{n \to \infty} (b_n)^{1/n} = 2 \) then \( \lim_{n \to \infty} (c_n)^{1/n} = 3 \).

Theorem 3. \( \lim_{n \to \infty} (b_n)^{1/n} = 2 \) if \( \lim_{n \to \infty} (c_n)^{1/n} = 3 \).

Table 1: Values of \( a_n, b_n, c_n \) for small \( n \).

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\( a_n \) has the interesting pattern in the table that for even \( n \), \( a_n = c_{n+1} \). We conjecture this holds always.

Conjecture 4. For all even \( n \), \( a_n = c_{n+1} \).

Consider the next theorem that is different from Conjecture 1.

Conjecture 5. For all even \( n \), \( a_n = c_{n+1} \).

References


For Further Information

• Email jpeebles@hmc.edu.

• See http://www.math.hmc.edu/seniorthesis/archives/2013/