Two-dimensional examples

Rectangular oscillating membrane

\[
\begin{align*}
& a := \pi : b := \pi : c := 1 : \\
& u := (m, n, x, y, t) \rightarrow \\
& \quad 16 \ast ((-1)^{m-1} \ast (-1)^{n-1}) / n^3 / m^3 / \pi^2 \ast \cos ((m^2 + n^2)^{1/2} \ast t) \ast \sin (m \ast x) \ast \sin (n \ast y) : \\
& \text{The first harmonic is periodic, with period } \sqrt{2} \pi : \\
& a \text{animate3d}(u(1,1,x,y,t), x=0..a, y=0..b, t=0..\pi * \sqrt{2}, \\
& \quad \text{axes=framed, thickness=2, frames=16);}
\end{align*}
\]

\[
\text{u}(1, 3, x, y, t) \text{ has period } \frac{2 \pi}{\sqrt{10}} :
\]
\[
\text{animate3d}(u(1, 3, x, y, t), x=0..a, y=0..b, t=0..2\pi/sqrt(10), \text{axes=framed, thickness=2, frames=16});
\]

\[
u(5, 3, x, y, t) \text{ has period } \frac{2\pi}{\sqrt{34}}.
\]
Since the various harmonics have different frequencies, a linear combination will not be periodic, but *almost periodic.*
combination := u(1,1,x,y,t) + 10*u(1,3,x,y,t) +
200*u(5,3,x,y,t):

animate3d(combination, x=0..a, y=0..b, t=0..1.35*Pi, axes=framed, thickness=2, frames=16);
This is a round membrane:

```
> restart: with(plots):
> upartic:=(r, theta, t) ->
   3.583422770*3^(1/2)/((1.913229428*Pi+105.4984657)*Pi)^(1/2)*cos
   (2.567811151*t)*BesselJ(2, 5.135622302*r)*sin(2*theta):
> Tperiod := 2*Pi/2.567811151:
> addcoords(z_cylindrical,[z,r,theta],
   [r*cos(theta), r*sin(theta), z]):
> animate3d(upartic(r,theta,t),r=0..1,theta=0..2*Pi,t=0..Tperiod,
   coords=z_cylindrical,axes=BOXED, frames = 12);
```
Damped string

Let us consider a homogeneous problem corresponding to the damped string:

PDE: \( u_{tt} = c^2 u_{xx} - \gamma u_t \), where \( c^2 = \frac{T}{\rho} \) and \( \gamma \) is a damping factor (positive);

BC: \( u(0, t) = 0 \),
\( u(l, t) = 0 \);

IC: \( u(x, 0) = f(x) \),
\( u_t(x, 0) = g(x) \).

We seek non trivial solutions (eigenfunctions), using the method of separation of variables,
\( u_n(x, t) = X(x) T(t) \).

As before, the given initial conditions yield
\( X_n(x) = \sin \left( \frac{n \pi x}{l} \right), n = 1, 2, ... \)

The time factor is solved next, giving
\( T_n(t) = e^{\frac{-\gamma t}{2}} \left( A_n \cos(\alpha_n t) + \sin(\alpha_n t) \right) \),
where
\( \alpha_n = \sqrt{\frac{c_n \pi}{l}} - \frac{\gamma^2}{4} \),
assuming that \( \gamma \) is sufficiently small, so the expression under the radical is positive
for all \( n = 1, 2, ... \)

The general solution of the whole problem (including the initial conditions) is sought as a linear combination of the eigenfunctions.

Let us consider a particular case.

\[
\begin{align*}
\text{restart:with plots):} \\
\text{Warning, the name changecoords has been redefined} \\
\text{1 := 1: c := 1: rho := 1: T := 1:} \\
\text{unprotect(gamma): gamma := 2:} \\
\text{assume(n::integer);} \\
\text{alpha := n -> sqrt((c*n*Pi/l)^2 - gamma^2/4):} \\
\text{Initial position:} \\
\text{f := x -> x*(1-x):} \\
\text{Initial velocity:} \\
\text{g := x -> 0:} \\
\text{A := n -> (2/l)*int(f(x)*sin(n*Pi*x/l), x=0..1):} \\
\text{B := n -> (2/alpha(n)/l)*int((gamma*f(x)/2+g(x))*} \\
\text{sin(n*Pi*x/l), x=0..1):} \\
\text{Eigenfunctions:}
\end{align*}
\]
\[ \text{part} := (n, x, t) \rightarrow e^{-\gamma t/2} \left( A(n) \cos(\alpha(n) t) + B(n) \sin(\alpha(n) t) \right) \sin(n\pi x/l) : \]

\[
\text{animate}(\text{part}(1, x, t), x=0..1, t=0..4, \text{frames}=36, \text{thickness}=2);
\]

\[
\text{plot3d}(\text{part}(1, x, t), x=0..1, t=0..4, \text{orientation}=[-15, 77]);
\]
The plucked string

[> restart: with(plots):

PDE: \( u_{tt} = c^2 u_{xx} \)

BC: \( u(0, t) = 0, \ u(L, t) = 0. \)

IC: \( u(x, 0) = f(x), \ u_t(x, 0) = 0. \)

Eigenvalues:

[> lambda := n -> n*pi/l:

Initial position: \( f(x) \) is a triangle (plucked string).
We assume that at a point \( p \) on \( (0, l) \) the string is lifted to height \( h \), remaining fixed at the endpoints.

[> assume(0<p, p<l, 0<h):
[> f := x -> piecewise(x <=p, h*x/p, x > p, h*(l-x)/(l-p)):

We plot the function for particular values of the parameters:

[> particular := \{l=1, h=1/2, p=2/3, c=1\}:
[> plot(subs(partial,f(x)), x=0..1, y=0..1);

\[ \text{Fourier coefficients, } A(n). \]

[> Af := n -> (2/l)*int(f(x)*sin(lambda(n)*x), x=0..1):
[> result := subs({cos(n*pi)=(-1)^n, sin(n*pi)=0}, Af(n)):
[> A := unapply(result, n):

We find the amplitudes for particular values of the parameters:

[> AA := subs(partial, unapply(result, n));
This is the solution (eigenfunction) for the particular values we chose:

\[ u := (n, x, t) \rightarrow (A_A(n) \cdot \cos(c \cdot \lambda(n) \cdot t)) \cdot \sin(\lambda(n) \cdot x) : \]

Approximate solution, for our particular choice of the parameters:

\[ u_a := (x, t) \rightarrow \text{subs(particular, sum(u(n,x,t), n=1..17))} : \]

\[ \text{animate}(u_a(x,t), x=0..1, t=0..2, color=red, thickness=2); \]

This is a plot of the surface $u(x, t)$.

\[ \text{plot3d}(u_a(x,t), x=0..1, t=0..2); \]
Localized plucking

Localized plucking, to better see the traveling waves.

We give particular values to all parameters:

\[
\begin{align*}
& l := 1; \quad h := 1/2; \quad p := 2/3; \quad c := 1; \\
& a := 2/3 - 0.2; \quad b := 2/3 + 0.1; \\
& f_{\text{small}} := x \to \begin{cases} \\
& x \geq a \text{ and } x \leq p, h^* (x-a)/(p-a), \\
& x \geq p \text{ and } x \leq b, h^* (x-b)/(p-b) \\
& \end{cases}; \\
& \text{plot}(f_{\text{small}}(x), x=0..1, y=0..1); \\
\end{align*}
\]
\textbf{Approximate solution for this case:}

\begin{verbatim}
> Asmall := m -> (2/l)*int(fsmall(x)*sin(lambda(m)*x), x=0..l):
> usmall := (n, x, t) -> (Asmall(n)*cos(c*lambda(n)*t))
    *sin(lambda(n)*x):

> uaa := (x, t) -> sum(usmall(n, x, t), n=1..23):
> animate(uaa(x, t), x=0..1, t=0..2, color=red, thickness=2, frames=22);
\end{verbatim}

This is a plot of the surface $u(x, t)$.

\begin{verbatim}
> plot3d(uaa(x, t), x=0..1, t=0..2, grid=[60,60]);
\end{verbatim}
Musical instruments

We have already seen the plucked string. Let us discuss other problems for the string equation, arising from the way musical instruments are played.

Localized impulse

If we hit the string with an impulse \( K \) concentrated at a point \( p \), (say we hit the string with the blade of a knife) then the solution is given by

The \( n \)th armonic is given by

\[
\text{uh} := (n, x, t) \rightarrow \frac{2K}{(\pi c \rho)} \frac{1}{n} \sin(\pi n p/l) \sin(\pi n x/l) \sin(\pi n c t/l);
\]

\[
\text{particular} := \{ l=1, c=1, K=1, p=2/3, \rho = 1 \}:
\]

Approximate solution to the "impulse start":

\[
\text{uimp} := (x,t) \rightarrow \sum_{n=1}^{9} \text{uh}(n,x,t);
\]

\[
\text{animate(subs(particular,uimp(x,t))), x=0..1, t=0..2, thickness=2);}
\]

\[
\text{plot3d(subs(particular,uimp(x,t))), x=0..1, t=0..2, axes=BOXED);}
\]
Small flat hammer

\[
\text{restart: with(plots):}
\]

The initial position is zero, the initial velocity is constant, equal to \( v_0 \), on a small interval \((p - \delta, p + \delta)\). The solution is given by

\[
> u(x, t) := \text{Sum}(u[n](x, t), n = 1..\infty):
\]

where the nth harmonic is given by

\[
> u[n] := (n, x, t) \rightarrow 4*\text{v}[0]*l/(\Pi^2*c)*(1/n^2)*
\]
\[
\sin(\Pi*n*p/l)*\sin(\Pi*n*delta/l)*
\]
\[
\sin(\Pi*n*x/l)*\sin(\Pi*n*c*t/l):
\]

\[
> \text{particular} := \{l=1, c=1, v[0]=1, p=2/3, rho = 1, delta=.1\}:
\]

Approximate solution to the "impulse start":

\[
> \text{uimp} := (x, t) \rightarrow \text{sum}(u[n](x, t), n = 1..9):
\]

\[
> \text{animate}(\text{subs}(\text{particular, uimp(x, t)}), x = 0..1,
\]
\[
t = 0..2, \text{thickness}=2);
\]

\[
> \text{plot3d}(\text{subs}(\text{particular, uimp(x, t)}), x = 0..1, t = 0..2,
\]
\[
\text{axes=BOXED});
\]