PDE’s & Maple Lab 1

1 Easy

You can download a MAPLE worksheet that does these problems from:

http://www.math.hmc.edu/~ajb/PCMI/PDE_Lab1.mws

1. Consider the convection equation

\[ u_t + cu_x = 0, \quad x \in \mathbb{R}, t > 0, \]  

and let \( F(x) = e^{-x^2} \).

(a) Show \( u(x, t) = F(x - ct) \) solves (1).

(b) Let \( c = 1 \). Plot \( u(x, t) \) at time \( t = 0, 1, 2 \) on the same axes.

(c) Plot the solution surface \( u(x, t) \) for \(-6 < x < 6 \) and \( 0 \leq t < 2 \).

(d) Create an animation of the solution \( F(x - t) \) for \( 0 < t < 2 \).

2. Consider Laplace’s equation in \( \mathbb{R}^2 \):

\[ \Phi_{xx} + \Phi_{yy} = 0, \quad (x, y) \in \mathbb{R}^2, \]  

and let \( F(x, y) = e^{-x} \cos y \).

(a) Show \( F \) solves (2).

(b) Plot \( F(x, y) \) for \( 0 < x < 1 \) and \( 0 < y < 2\pi \).

3. Consider the function \( f(x) = x \) on the interval \([0, \pi]\). The Fourier sine series for \( f \) is given by

\[ f(x) = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin mx. \]  

Plot the function \( f \) together with its Fourier approximation taking 2, 4, 8, 16 and then 32 terms of the series.
2 Medium

1. Consider the wave equation

\[ u_{tt} = c^2 u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \]

and let \( F(x) = e^{-x^2} \).

(a) Show \( u(x, t) = \frac{1}{2} \left( F(x - ct) + F(x + ct) \right) \) solves (4).

(b) Let \( c = 1 \). Plot \( u(x, t) \) at time \( t = 0, 1, 2 \) on the same axes.

(c) Plot the solution surface \( u(x, t) \) for \(-6 < x < 6 \) and \( 0 \leq t < 2 \).

(d) Create an animation of the solution \( u(x, t) \) for \( 0 < t < 2 \).

2. Consider the heat equation

\[ u_t = u_{xx}, \quad x \in \mathbb{R}, \ t > 0, \]

and let \( u(x, t) = \frac{1}{\sqrt{4\pi(t+1)}} e^{-\frac{x^2}{4(t+1)}} \).

(a) Show \( u(x, t) \) solves (5).

(b) Plot \( u(x, t) \) at time \( t = 0, 1, 2 \) on the same axes.

(c) Plot the solution surface \( u(x, t) \) for \(-6 < x < 6 \) and \( 0 \leq t < 2 \).

(d) Create an animation of the solution \( u(x, t) \) for \( 0 < t < 2 \).

(e) Show that \( \int_{-\infty}^{\infty} u(x, t) \, dx = 1 \) for each \( t > 0 \).

3. Consider the following initial boundary value problem for the heat equation:

\[ \begin{cases} u_t = u_{xx}, & x \in (0, \pi), \ t > 0, \\ u(0, t) = u(\pi, t) = 0, & t > 0, \\ u(x, 0) = x, & x \in [0, \pi]. \end{cases} \]

The Fourier series solution to this is given by

\[ u(x, t) = \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{m} e^{-m^2t} \sin mx. \]

Plot the 16 term approximation to the solution \( u(x, t) \) at time \( t = 0, 1, 2 \). Create an animation of the solution \( u(x, t) \) for \( 0 < t < 4 \).
3 Challenge

1. Harmonic Polynomials.
   (a) Show $F(x, y) = x^3 - 3xy^2$ is harmonic on $\mathbb{R}^2$. Plot $F$.
   (b) Find all cubic harmonic polynomials, i.e., all harmonic polynomials of the form $H(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$.

2. d’Alembert’s Solution. Let $u(x, t)$ be defined by

   $$u(x, t) = \frac{1}{2} \int_{x-t}^{x+t} e^{-s^2} ds.$$  \hspace{1cm} (7)

   (a) Show $u(x, t)$ solves the wave equation $u_{tt} = u_{xx}$. What is the initial displacement and velocity?
   (b) Plot $u(x, t)$ at time $t = 0, 1, 2$ on the same axes.
   (c) Plot the solution surface $u(x, t)$ for $-6 < x < 6$ and $0 \leq t < 2$.
   (d) Create an animation of the solution $u(x, t)$ for $0 < t < 2$.

3. The Erf Function. Let $v(x)$ be defined by

   $$v(x) = \int_0^x e^{-s^2} ds.$$  \hspace{1cm} (8)

   (a) Show $u(x, t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} v\left( \frac{x}{\sqrt{4t}} \right)$ solves the heat equation $u_t = u_{xx}$.
   (b) Plot $u(x, t)$ at time $t = .5, 1, 2$ on the same axes. What is the initial temperature distribution?
   (c) Plot the solution surface $u(x, t)$ for $-6 < x < 6$ and $0 \leq t < 2$.
   (d) Create an animation of the solution $u(x, t)$ for $0 < t < 2$.
   (e) Show $u_x(x, t)$ also solves the heat equation. Find and plot this solution. What is its initial temperature distribution?