

Some Random Problems from Probability

This problem session is modelled after the HMC Putnam Preparation Problem Solving Seminar co-led with Francis Su. Additional resources (and this problem set) can be found at:

http://www.math.hmc.edu/~ajb/PCMI/problem_solve.html

A1: A coin is dropped on a floor which consists of square tiles (in a checkerboard pattern) with a side length exactly equal to the diameter of the quarter. What is the probability that the coin covers the corner of some tile?

A2: Three points are chosen at random on a circle, and connected to form a triangle. What is the probability that the center of the circle is contained in the interior of the triangle?

A3: A square matrix has entries that are chosen randomly to be either 0 or 1.

a) If the matrix is 2×2 , what is the probability that the determinant of the matrix vanishes?

b) Suppose the matrix is 3×3 . What is the probability now?

c) (Very Hard) Suppose the matrix is $n \times n$. What can you say about the probability now?

A4: A knight confined to a 5×5 chessboard instantaneously makes standard knight's moves each second in such a way that it is equally likely to move to any of the squares one move away from it. What long-run fraction of the time does it occupy the center square? (Hess)

A5: You're one of a hundred people standing in line to get onto an airplane that has 100 seats. There's a seat for every person who's in line, and each of you has a boarding pass for your assigned seat. The first person to walk onto the plane drops his boarding pass and, instead of picking it up, decides, "I'm just going to sit anywhere." He takes a seat at random.

Now, every other passenger will take either his assigned seat or, if that seat is taken, that passenger will take any seat at random.

Because you are such a kind, generous, and accommodating person, you are the last passenger to walk onto the plane. Obviously, there's going to be one seat left.

What are the chances that you get to sit in your assigned seat? (Car Talk Puzzler)

And for a little bit of variety...

A6: A not so random walk: A mathematician gets lost in the woods. She knows its area is 1 square mile, but knows nothing else about the shape except that it has no holes in it (i.e. it is simply-connected).

a) Show that she can escape by walking less than 3.6 miles.

b) Show she can escape by walking less than 2.6 miles if she knows the woods is convex.

Hints:

1. Consider the location of the center of the circle.
2. Can you identify the points on the circle with a $2\pi \times 2\pi$ square?
3. It may help to think about the columns of the matrix. What do we know if the determinant is non-zero?
4. By symmetry, some of the squares have identical occupation probabilities. How can you use this to simplify the problem?
5. Amazingly, the answer is independent of the number of seats in the plane. What happens with two seats? Three seats?
6. What region has the smallest perimeter for a given area?