Magic “Squares” Indeed!

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1 Introduction

Behold the remarkable property of the magic square:

\[
\begin{bmatrix}
6 & 1 & 8 \\
7 & 5 & 3 \\
2 & 9 & 4 \\
\end{bmatrix}
\]

\[618^2 + 753^2 + 294^2 = 816^2 + 357^2 + 492^2 \text{ (rows)}\]
\[672^2 + 159^2 + 834^2 = 276^2 + 951^2 + 438^2 \text{ (columns)}\]
\[654^2 + 132^2 + 879^2 = 456^2 + 231^2 + 978^2 \text{ (diagonals)}\]
\[639^2 + 174^2 + 852^2 = 936^2 + 471^2 + 258^2 \text{ (counter-diagonals)}\]
\[654^2 + 798^2 + 213^2 = 456^2 + 897^2 + 312^2 \text{ (diagonals)}\]
\[693^2 + 714^2 + 258^2 = 396^2 + 417^2 + 852^2 \text{ (counter-diagonals)}\]

This property was discovered by Dr. Irving Joshua Matrix [3], first published in [5] and more recently in [1]. We prove that this property holds for every 3-by-3 magic square, where the rows, columns, diagonals, and counter-diagonals can be read as 3-digit numbers in any base. We also describe n-by-n matrices that satisfy this condition, among them all circulant matrices and all symmetrical magic squares. For example, the 5-by-5 magic square in (1) also satisfies the square-palindromic property for every base.

\[
\begin{bmatrix}
17 & 24 & 1 & 8 & 15 \\
23 & 5 & 7 & 14 & 16 \\
4 & 6 & 13 & 20 & 22 \\
10 & 12 & 19 & 21 & 3 \\
11 & 18 & 25 & 2 & 9 \\
\end{bmatrix}
\]

We must be careful when we read these numbers. The base 10 number represented by the first row of (1) is \(17 \cdot 10^4 + 24 \cdot 10^3 + 1 \cdot 10^2 + 8 \cdot 10 + 15 = 194195\). The base 10 number based on the first row’s reversal is 158357.
2 Sufficient Conditions

We say that a real matrix is square-palindromic if, for every base $b$, the sum of the squares of its rows, columns, and four sets of diagonals (as in the previous examples) are unchanged when the numbers are read “backwards” in base $b$. We can express this condition using matrix notation. Let $M$ be an $n$-by-$n$ matrix. Then the $n$ numbers (in base $b$) represented by the rows of $M$ are the entries of the vector $Mb$, where $b = (b^{n-1}, b^{n-2}, \ldots, b, 1)^T$, and $T$ denotes the transpose operation. The sum of the squares of these numbers is

$$(Mb)^T Mb = b^T (M^T M) b.$$ 

Next, the $n$ numbers represented by the rows when read “backwards” are the entries of $M^R b$ where the $n$-by-$n$ reversal matrix $R = [r_{ij}]$ has $r_{ij} = 1$ if $i + j = n + 1$, and $r_{ij} = 0$ otherwise. Note that $R^T = R^{-1} = R$. The sum of the squares of these numbers is

$$(M^R b)^T (M^R b) = b^T (R (M^T M) R) b.$$ 

Hence a sufficient condition for the rows of $M$ to satisfy the square-palindromic property is simply $R (M^T M) R = M^T M$. Matrices $A$ that satisfy $RAR = A$ are called centro-symmetric, [6]: $a_{ij} = a_{n+1-i,n+1-j}$. Matrices $A$ that satisfy $RAR = A^T$ are called persymmetric, [4]: $a_{ij} = a_{n+1-j,n+1-i}$. It is easy to see that symmetric matrices that are centro-symmetric must also be persymmetric. Since $M^T M$ is necessarily symmetric, our sufficient condition says that $M^T M$ is centro-symmetric, or equivalently, that

$M^T M$ is persymmetric.

The square-palindromic condition for the columns of $M$ is the square-palindromic condition for the rows of $M^T$. Hence it suffices to require that

$M^T M$ is persymmetric.

For the first set of diagonals, we create a matrix $\tilde{M}$ with the property that each column of $\tilde{M}$ represents a diagonal starting from the first row of $M$. To do this, we introduce two other special square matrices. Let $P_k = [p_{ij}]$ denote the $n$-by-$n$ projection matrix whose only non-zero entry is $p_{kk} = 1$. Notice that $P^T = P$, and $P_k M$ preserves the $k$-th row of $M$ but turns all other rows to zeros. Let $S = [s_{ij}]$ denote the $n$-by-$n$ shift operator where $s_{ij} = 1$ if $i - j \equiv 1$ (mod $n$), $s_{ij} = 0$ otherwise.

The following properties of $S$ are easily verified: $S^n = I_n$, $S^{-1} = S^T = RSR$, and $MS^k$ shifts the columns of $M$ over “$k$ steps to the left”. Now define

$$\tilde{M} = \sum_{i=1}^n P_i M S^{i-1}.$$
Hence the $i$-th diagonal of $M$, starting from the first row becomes the $i$-th column of $\tilde{M}$. By the column condition, these diagonals satisfy the square-palindromic property if the $(i, j)$ entry of $\tilde{M}M^T$ equals its $(n + 1 - j, n + 1 - i)$ entry.

We have

$$\tilde{M}M^T = \sum_{i=1}^{n} P_i MS^{i-1} \left( \sum_{j=1}^{n} P_j MS^{j-1} \right)^T$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} P_i MS^{i-j} M^T P_j.$$  

It follows that $\tilde{M}M^T$ has the same $(i, j)$ entry as $MS^{i-j}M^T$, and the same $(n + 1 - j, n + 1 - i)$ entry as well; if $MS^{i-j}M^T$ is persymmetric, then these entries are equal. Consequently, these diagonals obey the square-palindromic property if

$$MS^k M^T$$

is persymmetric for $k = 1, \ldots, n$.  

(2)

Conveniently, (2) also ensures that the counter-diagonals starting from the first row satisfy the square-palindromic property. This can be seen by mimicking the preceding explanation with $\tilde{M} = \sum_{i=1}^{n} P_i MS^{-(i-1)}$, whereby $\tilde{M}M^T$ has the same $(i, j)$ and $(n + 1 - j, n + 1 - i)$ entry as $MS^{i-j}M^T$. For the other diagonal and counterdiagonal, we obtain similar results [7], which we summarize in the following theorem:

**Theorem 1** A square matrix $M$ has the square-palindromic property if the following matrices are all persymmetric:

1. $M^T M$,
2. $MM^T$,
3. $MS^k M^T$, for $k = 1, \ldots, n$, and
4. $M^T S^k M$, for $k = 1, \ldots, n$.

**3 Square-Palindromic Matrices**

Next we explore classes of matrices that are square-palindromic. We say that a square matrix $A$ is *centro-skew-symmetric* if $RAR = -A$, that is, $a_{ij} + a_{n+1-i,n+1-j} = 0.$
Theorem 2  Every centro-symmetric or centro-skew-symmetric matrix is square-palindromic.

Proof: If $M$ is centro-symmetric or centro-skew-symmetric, then the relations $RM = \pm MR$ and $R(S^k)R = S^{-k}$ ensure that $M$ satisfies the conditions of Theorem 1. □

The theorem is not at all surprising since the collection of rows, columns and diagonals of $M$ read the same backwards and forwards. The next class of matrices, however, satisfies the conditions in a non-obvious way.

We say that $A$ is **circulant** if every entry of each “diagonal” is the same, i.e., $a_{ij} = a_{k\ell}$ if $i - j \equiv k - \ell \mod n$, or simply $SAS^{-1} = A$. We say that $A$ is **(-1)-circulant** if $SAS = A$.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
4 & 1 & 2 & 3 \\
3 & 4 & 1 & 2 \\
2 & 3 & 4 & 1
\end{bmatrix}
\]

Circulant

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
3 & 4 & 5 & 1 & 2 \\
4 & 5 & 1 & 2 & 3 \\
5 & 1 & 2 & 3 & 4
\end{bmatrix}
\]

(-1)-Circulant

Notice that the circulant and (-1)-circulant property is preserved under transposing. It is easy to show that the product of two circulant matrices or two (-1)-circulant matrices is circulant, while the product of a circulant and (-1)-circulant matrix is (-1)-circulant. Note that $S$ is circulant, $R$ is (-1)-circulant, and that all circulant matrices are persymmetric since $a_{ij}$ and $a_{n+1-j,n+1-i}$ lie on the same diagonal. Consequently, if $M$ is circulant or (-1)-circulant, the matrices $M^T M$, $M M^T$, $S^k M^T$, and $M^T S^k M$ are all circulant, and thus persymmetric. From Theorem 1, it follows that

Theorem 3  Every circulant or (-1)-circulant matrix is square-palindromic.

Notice that four of the six square-palindromic identities are not obvious, but two of the diagonal sums are completely trivial!

4  Magic and Semi-Magic Squares

A **semi-magic square** with magic constant $c$ is a square matrix $A$ in which every row and column adds to $c$. Using matrix notation, this says that $AJ = cJ = JA$, where $J$ is the matrix of all ones. If the main diagonal and main counter-diagonal also add to $c$, then the matrix is called a **magic square**. Circulant and (-1)-circulant matrices are always semi-magic, but are not necessarily magic.

A magic square $A$ is **symmetrical** [2] if the sum of each pair of two entries that are opposite with respect to the center is $2c/n$, that is $a_{ij} + a_{n+1-j,n+1-i} = 2c/n$. Notice that a semi-magic square with this property is magic.
Like the example below, magic and semi-magic squares do not necessarily satisfy the square-palindromic property.

\[
\begin{bmatrix}
2 & 0 & 1 \\
0 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

Semi-Magic but not square-palindromic

However,

**Theorem 4** Every symmetrical magic square is square-palindromic.

**Proof:** The trick is to notice that if \( M \) is a symmetrical magic square with magic constant \( c \), then \( M = M_0 + cJ/n \), where \( M_0 \) is a symmetrical magic square with magic constant 0. But this implies that \( M_0 \) is centro-skew-symmetric. Therefore \( M_0 \) is square-palindromic and satisfies the conditions of Theorem 1. Thus, since \( M_0^T \) and \( J \) are persymmetric, it follows that \( M^T M = (M_0 + cJ/n)^T (M_0 + cJ/n) = M_0^T M_0 + c^2 J/n \) is also persymmetric. Hence \( M \) satisfies condition 1 of Theorem 1. To verify condition 3 (the other cases are similar), notice that

\[
MS^kM^T = (M_0 + \frac{c}{n}J)S^k(M_0 + \frac{c}{n}J)^T
\]

\[
= M_0S^kM_0^T + \frac{c^2}{n}J
\]

is persymmetric for \( k = 1, \ldots, n \), since \( M_0 \) satisfies condition 3 of Theorem 1. 

\( \square \)

Although not all magic squares are square-palindromic, it is easy to see that all 3-by-3 magic squares are symmetrical. Consequently, we have

**Theorem 5** All 3-by-3 magic squares are square-palindromic.

**References**


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