Random Approaches to Fibonacci Identities

Arthur T. Benjamin; Gregory M. Levin; Karl Mahlburg; Jennifer J. Quinn


Stable URL:
http://links.jstor.org/sici?sici=0002-9890%28200006%2F07%29107%3A6%3C511%3ARATFI%3E2.0.CO%3B2-B

*The American Mathematical Monthly* is currently published by Mathematical Association of America.

Your use of the JSTOR archive indicates your acceptance of JSTOR’s Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR’s Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/maa.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact support@jstor.org.
Random Approaches to Fibonacci Identities

Arthur T. Benjamin, Gregory M. Levin, Karl Mahlburg, and Jennifer J. Quinn

There is no problem in all mathematics that cannot be solved by direct counting.
—Ernst Mach

1. INTRODUCTION. Many combinatorialists live by Mach’s words, and take it as a personal challenge. For example, nearly all of the Fibonacci identities in [5] and [6] have been explained by counting arguments ([1], [2], [3]). Among the holdouts are those involving infinite sums and irrational quantities. However, by adopting a probabilistic viewpoint, many of the remaining identities can be explained combinatorially. As we demonstrate, even the irrational-looking Binet formula for the $n$-th Fibonacci number

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]$$

(1)

can be rationalized by this approach.

We introduce several models for generating random tilings that yield fundamental identities for Fibonacci numbers and their generalizations. The models share the feature that the probability of every length $n$ tiling depends only on $n$.

We begin by recalling a combinatorial interpretation of Fibonacci numbers. Let $c_n$ denote the number of series of 1’s and 2’s that sum to $n$. Then $c_n = F_{n+1}$ since $c_1 = 1 = F_2$, $c_2 = 2 = F_3$, and (by conditioning on the first number in the series) $c_n = c_{n-1} + c_{n-2}$. Thus, for $n \geq 1$, $F_n$ may be combinatorially defined as the number of ways to tile a board of length $n - 1$ using squares and dominoes.

2. BLACK AND WHITE MODEL. For the first model, consider an infinite board with cells 1, 2, …, and color each cell black or white, independently, with probability $1/2$. See Figure 1 for an example. Any coloring of the first $n$ cells has probability $(1/2)^n$.

An infinite tiling can be viewed as alternating strings of black and white cells of varying lengths. For example, the tiling in Figure 1 has a black string of length 4 followed by a white string of length 3 followed by a black string of length 3 followed by a white string of length 1, and so on.

Let the random variable $X$ denote the location of the end of the first black string of odd length. In our example, $X = 10$. Now we ask the question “For $n \geq 1$, what is the probability that $X = n?$” To answer this, observe that a tiling has $X = n$ if and only if cell $n$ is covered by a black square, cell $n + 1$ is covered
by a white square, and cells 1 through \( n - 1 \) can be covered by white squares and black dominoes. There are \( F_n \) ways to tile the first \( n - 1 \) cells in this manner; therefore, \( P(X = n) = F_n / 2^{n+1} \). Since \( X \) is finite with probability 1, this gives a combinatorial explanation of the identity

\[
\sum_{n \geq 1} F_n / 2^{n+1} = 1.
\]

A similar strategy allows us to explain

\[
\sum_{n \geq 1} nF_n / 2^{n+1} = 5.
\]

From the preceding discussion, the left side of this identity evaluates \( E(X) \), the expected value of \( X \). To prove that \( E(X) = 5 \), we express \( X \) as the sum of three random variables,

\[
X = B + 2D + R.
\]

\( B \) is a geometric random variable with mean 2; it denotes the location of the first black square. \( D \) is the number of black dominoes, i.e., pairs of black squares, that immediately follow the first black square. Here \( D + 1 \) is a geometric random variable with mean \( 4/3 \). Finally, \( R \) denotes the remaining number of white and black tiles needed, possibly zero, to obtain the end of the first black string of odd length. (For the tiling in Figure 1, \( B = 1, D = 1, \) and \( R = 7 \).) The colors of the two squares that cover cells \( B + 2D + 1 \) and \( B + 2D + 2 \) are, with equal probability, white-white, white-black, or black-white. In the first two cases, \( R = 0 \), since cell \( B + 2D \) is the end of a black string of odd length. In the third case, cell \( B + 2D + 1 \) is the end of a black string of even length, cell \( B + 2D + 2 \) has a white square, and we are back to the drawing board. Thus by the linearity of expectation,

\[
E(X) = E(B) + 2E(D) + E(R)
\]

\[
= 2 + 2 \left( \frac{4}{3} - 1 \right) + \frac{2}{3}(0) + \frac{1}{3}(2 + E(X)). \tag{2}
\]

Solving equation (2) gives us \( E(X) = 5 \) as desired.

The preceding identities can also be proved using a different combinatorial model, suggested to us by Stephen Maurer. It is easy to show that \( F_n \) counts the number of binary sequences of length \( n - 2 \) with no consecutive zeros. By considering a random infinite binary string, and letting \( X \) denote the beginning of the first double zero, one again obtains \( P(X = n) = F_n / 2^{n+1} \) and \( E(X) = 5 \).

3. RANDOM APPROACHES TO BINET’S FORMULA. Binet’s formula was first established by Abraham De Moivre in 1718, using generating functions. It was independently rediscovered by Jacques Binet (1843) and Gabriel Lamé (1844). For our first proof of Binet’s formula, we tile an infinite board by independently placing squares and dominoes, one after another. At each decision, we use a square with probability \( 1/\phi \) or a domino with probability \( 1/\phi^2 \), where \( \phi = (1 + \sqrt{5})/2 \approx 1.618 \). Conveniently, \( 1/\phi + 1/\phi^2 = 1 \). A random example is shown in Figure 2. In this model, the probability that a tiling begins with any particular length \( n \) sequence of squares and dominoes is \( 1/\phi^n \).

![Figure 2. A random tiling of squares and dominoes.](image-url)
We apply this model to derive Binet’s formula. Since \((1 - \sqrt{5})/2 = -1/\phi\), Binet’s formula (1) says

\[
F_n = \frac{1}{\sqrt{5}} \left[ \phi^n - (-1/\phi)^n \right].
\]  (3)

Let \(q_n\) be the probability that a random tiling is breakable at cell \(n\), i.e., that a square or domino begins at cell \(n\). The example in Figure 2 is breakable at cells 1, 3, 4, 6, 8, 9, 11, and so on. Since there are \(F_n\) different ways to tile the first \(n - 1\) cells,

\[
q_n = F_n / \phi^{n-1}.
\]  (4)

For a tiling to be unbreakable at \(n\), it must be breakable at \(n - 1\) followed by a domino. Thus for \(n \geq 2\),

\[
q_n = 1 - q_{n-1} / \phi^2,
\]  (5)

where \(q_1 = F_1 = 1\). Let \(q = \lim_{n \to \infty} q_n\). Taking a limit in (5) gives \(q = 1 - q / \phi^2\). Solving for \(q\) we find that \(q = (1 + 1/\phi^2)^{-1} = \phi / \sqrt{5}\).

Combined with (4), this gives us the asymptotic form of Binet’s formula:

\[
F_n \approx \phi^n / \sqrt{5}.
\]

To derive Binet’s formula exactly, simply unravel recurrence (5) along with initial condition \(q_1 = 1\) to get

\[
q_n = 1 - 1/\phi^2 + 1/\phi^4 - 1/\phi^6 + \cdots + (-1/\phi^2)^{n-1}.
\]  (6)

This is a finite geometric series that simplifies to

\[
q_n = \frac{\phi}{\sqrt{5}} \left[ 1 - \left( \frac{-1}{\phi^2} \right)^n \right].
\]  (7)

Multiplying both sides of (7) by \(\phi^{n-1}\) yields the desired identity.

Notice that by substituting (4) into (5) and multiplying by \(\phi^n\) we have also demonstrated

\[
\phi^n = \phi F_n + F_{n-1}.
\]

We briefly mention three other random approaches to Binet’s formula; for a discussion of more traditional approaches, see [4]. The process of randomly placing squares and dominos by the previous model can be thought of as a Markov chain that moves between breakable (B) and unbreakable (U) states according to the matrix of transition probabilities

\[
P = \begin{bmatrix}
B & U \\
1/\phi & 1/\phi^2 \\
\end{bmatrix}
\]

where \(p_{ij}\) is the probability of going from state \(i\) to state \(j\). We begin at time (cell) 1 in the breakable state. Thus \(q_n\), the probability that we are breakable at time \(n\),
is the \((1, 1)\) entry of \(P^{n-1}\). By diagonalization

\[
P^{n-1} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -\phi^2 & 0 & (-1/\phi^2)^{n-1} \\ 0 & (1 - \phi^2) & 1 & -\phi^2 \end{bmatrix}^{-1}
\]

whose \((1, 1)\) entry simplifies to (7).

Another approach is to create an “almost one-to-one” correspondence between tilings breakable at \(n\) and those breakable at \(n-1\). Suppose \(T\) is a tiling, breakable at \(n\), where \(n\) is odd. We describe a procedure for transforming \(T\) into a tiling that is breakable at \(n-1\). Define \(k\) to be the smallest number in \([2, \ldots, n]\) such that \(T\) is breakable at cells \(k\) and \(n + k - 2\). After swapping the tiles covering cells 1 through \(k - 1\) with those covering cells \(n\) through \(n + k - 3\) (as illustrated in Figure 3), \(T = a_1 a_2 b_1 b_2\) is transformed to \(T' = b_1 a_2 a_1 b_2\), which is breakable at \(n-1\). (When \(k = 2\) this shifts the tiles covering cells 2 through \(n-1\) to the left one cell, and moves the square on cell 1 to cell \(n-1\).) The procedure, when defined, is one-to-one, onto, and “preserves probability” since the string \(b_1 a_2 a_1\) has the same length as \(a_1 a_2 b_1\). The only strings for which the procedure is undefined are those that begin with \(n-1\) dominoes, which has probability \(1/\phi^{2(n-1)}\). Thus \(q_n = q_{n-1} + 1/\phi^{2(n-1)}\). By a similar argument, one can show that when \(n\) is even, \(q_n = q_{n-1} - 1/\phi^{2(n-1)}\). Combining the even and odd cases gives us

\[q_n = q_{n-1} + (-1/\phi^2)^{n-1},\]

which leads directly to (6).

![Figure 3](image.png)

**Figure 3.** An almost one to one correspondence between tilings breakable at \(n\) and tilings breakable at \(n-1\).

For an approach that produces (6) without the use of a recurrence, we use a different random tiling model. Initially label the cells of an infinite board with 1’s and 2’s, independently, with probability \(1/\phi\) and \(1/\phi^2\), respectively. From this labelling, create a tiling by reading from left to right, and replacing 1 with a square, and 21 or 22 with a domino. Thus the labelling 21121221212... generates (not uniquely) the tiling of Figure 2. As in the previous model, every tiling of length \(n\) has probability \(1/\phi^n\). Notice that a tiling is unbreakable at \(n\) if and only if cell \(n-1\) is labeled with 2 and the number (possibly zero) of 2’s that immediately precede it is even. For example, when \(n = 6\) the probability that the tiling is
unbreakable at cell 6 is

\[ 1 - q_6 = 1/\phi^2 - 1/\phi^4 + 1/\phi^6 - 1/\phi^8 + 1/\phi^{10}. \]

The first two terms compute the probability that our initial 5 labels are of the form abc12, where a, b, and c are arbitrary. The next two terms compute the probability of starting with a1222, where a is arbitrary, and the last term is the probability of beginning with 22222. Applying the same argument to an arbitrary \( n \) yields (6) directly.

4. BEYOND BINET. The combinatorial approach can be used to prove other Fibonacci-like identities. For example, the Lucas numbers 2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots satisfy:

\[ L_n = \phi^n + (-1/\phi)^n. \]  

(8)

As described in [1], \( L_n \) counts the number of ways to tile a length \( n \) board with squares and dominoes, with the caveat that if the tiling begins with a domino, the domino is assigned one of two different phases. In our random model, the first tile is either a square with probability \( 1/\sqrt{5} \), an in-phase domino with probability \( 1/\phi \sqrt{5} \) or an out-of-phase domino with probability \( 1/\phi^2 \sqrt{5} \). Thereafter, tiles are chosen at random and independently with probability \( 1/\phi \) for squares and \( 1/\phi^2 \) for dominoes. In this model, any length \( n \) tiling has probability \( 1/\phi^{n-1} \sqrt{5} \). Let \( r_n \) denote the probability that a tiling is breakable at \( n + 1 \). Thus for \( n \geq 2 \),

\[ r_n = L_n/\phi^{n-1} \sqrt{5}, \]

where \( r_1 = 1/\sqrt{5} \). By the same argument as in (5), for \( n \geq 2 \),

\[ r_n = 1 - r_{n-1}/\phi^2, \]

which unravels to

\[ r_n = 1 - 1/\phi^2 + 1/\phi^4 - 1/\phi^6 + \cdots + (-1/\phi^2)^{n-2} + (-1/\phi^2)^{n-1}/\sqrt{5}. \]

Summing the series results in

\[ r_n = \frac{\phi}{\sqrt{5}} \left[ 1 + \left( -\frac{1}{\phi^2} \right)^n \right], \]

which is the same as identity (8) with both sides divided by \( \phi^{n-1} \sqrt{5} \).

Similarly, any generalized Fibonacci sequence \( G_0, G_1, G_2, \ldots \), satisfying \( G_n = G_{n-1} + G_{n-2} \) for \( n \geq 2 \), can be shown to have closed form

\[ G_n = \alpha \phi^n + \beta (-1/\phi)^n, \]  

(9)

where \( \alpha = (G_1 + G_0 \phi)/\sqrt{5} \) and \( \beta = (\phi G_0 - G_1)/\sqrt{5} \); see [5] or [6].

When \( G_0 \) and \( G_1 \) are non-negative integers, \( G_n \) can be given the combinatorial interpretation of the number of ways to tile a length \( n \) board with squares and dominoes, where there are \( G_0 \) phases for an initial domino, and \( G_1 \) phases for an initial square [2].

In this random model, the first tile is either a square in one of \( G_1 \) phases with probability \( 1/(G_1 \phi + G_0) \) or a domino in one of \( G_0 \) phases with probability \( 1/(G_1 \phi + G_0) \). Thereafter, tiles are chosen at random and independently with probability \( 1/\phi \) for squares and \( 1/\phi^2 \) for dominoes. Here, any length \( n \) tiling has probability \( 1/(G_1 \phi + G_0) \phi^{n-2} \). Then for \( n \geq 2 \), unravelling the probability that a tiling is breakable at \( n \) allows (9) to be explained combinatorially.

REFERENCES


June–July 2000] RANDOM APPROACHES TO FIBONACCI IDENTITIES 515

**ARTHUR T. BENJAMIN** is Associate Professor of Mathematics at Harvey Mudd College in Claremont, California, where he has taught since 1989 after earning his Ph.D. from Johns Hopkins in Mathematical Sciences. His research interests are in combinatorics and game theory. He recently received the 2000 Deborah and Franklin Tepper Haimo Award for Distinguished Teaching from the MAA. *Harvey Mudd College, Claremont, CA 91711*

benjamin@hmc.edu

**GREGORY M. LEVIN** received a B.S. in mathematics from Harvey Mudd College in 1992 and a Ph.D. in mathematical sciences from The Johns Hopkins University in 1997. He has spent the last two years as a visiting professor at Harvey Mudd. His mathematical interests are in discrete mathematics, and the majority of his research has been in fractional graph theory. *Harvey Mudd College, Claremont, CA 91711*

levin@hmc.edu

**KARL MAHLBURG** is studying for a B.S. in mathematics at Harvey Mudd College, with expected graduation in 2001. His research interests include algebra, combinatorics, and topology, which he hopes to study more thoroughly at the graduate level.

*Harvey Mudd College, Claremont, CA 91711*

karl_mahlburg@hmc.edu

**JENNIFER J. QUINN** is Associate Professor of Mathematics at Occidental College in Los Angeles. She earned her B.A., M.S., and Ph.D. from Williams College, University of Illinois, University of Chicago, and University of Wisconsin.

*Occidental College, 1600 Campus Drive, Los Angeles, CA 90041*

jquinn@oxy.edu