Solitary waves and vortices

Vieri Benci

Dipartimento di Matematica Applicata “U. Dini”
Università di Pisa

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Roughly speaking a solitary wave is a solution of a field equation whose energy travels as a localized packet and which preserves this localization in time.

A soliton is a solitary wave which exhibits some strong form of stability so that it has a particle-like behavior.

We can define a vortex in a very general way as a soliton with a strong angular momentum.
Consider the following equation:

$$\Box \psi + W'(\psi) = 0$$  \hspace{1cm} (1)$$

where

$$\psi : \mathbb{R}^4 \to \mathbb{C}; \quad \Box = \frac{\partial^2}{\partial t^2} - c^2 \Delta$$

$$W'(\psi) = W'(|\psi|) \frac{\psi}{|\psi|}$$
Four types of waves

Depending on the form of $W$, the NWE presents different types of waves:

- Non-dispersive waves: D’Alembert equation
  \[ \Box \psi = 0 \]  \hspace{1cm} (2)

- Dispersive waves: Klein-Gordon equation
  \[ \Box \psi + \psi = 0, \]  \hspace{1cm} (3)

- Solitary waves: nonlinear equation
  \[ \Box \psi + \frac{\psi}{1 + |\psi|} = 0, \]  \hspace{1cm} (4)

- Vortices: the same nonlinear equations
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Dispersive waves: Klein-Gordon equation

Klein-Gordon

\[ \Box \psi + u = 0 \quad (7) \]
\[ \Box \psi + \frac{\psi}{1 + |\psi|} = 0 \]
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(9)
Soliton types

Today, we know (at least) three mechanism which might produce solitons:

- Complete integrability; e.g. Kortewg-de Vries equation
  \[ u_t + u_{xxx} + 6uu_x = 0 \]

- Topological constrains: e.g. Sine-Gordon equation
  \[ u_{tt} - u_{xx} + \sin u = 0 \]

- Ratio energy/charge: e.g. complex valued nonlinear wave equation (they exists also for \( N > 1 \))
  \[ \psi_{tt} - \Delta \psi + W'(\psi) = 0; \ \psi \in \mathbb{C} \]
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The hylomorphic solitons

In this talk, I will present part of the results obtained by our group in the last years on the third type of solitons.

In particular these results are due to: Badiale, Bellazzini, Benci, Bonanno, D’Aprile, D’Avenia, Fortunato, Ghimenti, Micheletti, Mugnai, Pisani, Rolando.

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I will refer to this type of solitons as

\[ \textit{hylomorphic solitons} \]
The case in which they are spherically symmetric has been largely studied in physics and they are called Q-balls.
Configurations of a charged scalar field that are classically stable were constructed by G. Rosen who first realized the possibility of stable solitary waves in the NWE.


Stable configurations of multiple scalar fields were studied by Friedberg, Lee, and Sirlin.

Hystory of Q-balls

The name "Q-ball" and the proof of quantum-mechanical stability come from Sidney Coleman


For applications to cosmology see also


Moreover, Q-balls have been studied also in gauge theories:

The main equations in which hylomorphic solitons may occur are the following:

Nonlinear Schrödinger equation:

$$\partial_t \psi = -\frac{1}{2} \Delta \psi + W'(\psi)$$

Nonlinear wave equation:

$$\Box \psi + W'(\psi) = 0$$
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\]

### Nonlinear wave equation:
\[
\Box \psi + W'(\psi) = 0
\]
Abelian gauge theory:

\[
D_t^2 \psi - \sum_{j=1}^{3} D_{x_j}^2 \psi + W'(\psi) = 0 \tag{10}
\]

\[
-\nabla \cdot \mathbf{E} = \rho(\psi, \varphi); \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0; \tag{11}
\]

\[
\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}(\psi, \mathbf{A}); \quad \nabla \cdot \mathbf{H} = 0; \tag{12}
\]
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\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} (\psi, \mathbf{A}); \quad \nabla \cdot \mathbf{H} = 0; \tag{12}\]

where

\[D_t = \partial_t + iq \varphi, \quad D_{x_j} = \partial_{x_j} - iq A_j,\]

\[
\rho (\psi, \varphi) = -q \text{ Im} (\partial_t \psi \overline{\psi}) + q^2 \varphi |\psi|^2 \tag{13}\]

\[
\mathbf{j} (\psi, \mathbf{A}) = q \text{ Im} (\nabla \psi \overline{\psi}) - q^2 \mathbf{A} |\psi|^2 \tag{14}\]
These equations satisfy the two assumptions which are shared by every fundamental theory in Physics.

A-1. They are variational.
A-2. They are invariant for the Poincaré (or the Galileo) group.

Moreover NWE satisfies also the following assumption:
A-3. It is invariant for the action of the group $S^1$: $T \phi \psi \mapsto e^{i \phi} \psi$.

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- **A-3.** It is invariant for the action of the group $S^1$:

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- **Energy.** Energy, by definition, is the quantity which is preserved by the time invariance of the Lagrangian; namely by the fact that it does not depend explicitly on $t$.

- **Charge.** The charge, by definition, is the quantity which is preserved by by the "gauge action" $\psi \rightarrow \psi e^{i\theta}$; namely by the fact that the Lagrangian does not depend explicitly on $\theta$. 

Now let us consider the NWE:

\[ \Box \psi + W' (\psi) = 0 \quad \text{(NWE)} \]

The solution of this equations are critical points of the functional

\[ J(\psi) = \int \int \left[ \frac{1}{2} |\partial_t \psi|^2 - \frac{1}{2} |\nabla \psi|^2 - W(\psi) \right] dx \, dt \]
Conservation laws for the NWE

- **Energy.** Energy which is the quantity which is preserved by the time invariance of the Lagrangian; in the NWE, it has the following form

\[ E = \int \left[ \frac{1}{2} |\partial_t \psi|^2 + \frac{1}{2} |\nabla \psi|^2 + W(\psi) \right] \, dx \]

- **Charge.** The charge which is the quantity which is preserved by the "gauge action" \( \psi \rightarrow \psi e^{i\theta} \); for the NWE, we get

\[ C = Im \int \partial_t \psi \bar{\psi} \, dx \]
Standing waves

The simplest kind solitary waves are the *standing waves* which, roughly speaking, are solitary waves which do not travel.
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A *standing wave* is a finite energy solution of NWE having the following form

\[
\psi(t, x) = u(x)e^{iS(t, x)}, \quad u \geq 0,
\]  

(15)
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\psi(t,x) = u(x)e^{iS(t,x)}, \quad u \geq 0,
\]  

(15)

Substituting (15) in NWE, we get

\[
-\Delta u + [k^2 - \omega^2]u^2 + W'(u) = 0
\]  

(16)

\[
\partial_t (\omega u^2) + \nabla \cdot (ku^2) = 0
\]  

(17)

where

\[
\omega = -\partial_t S; \quad k = \nabla S
\]
If $\psi(x) = \psi(x_1, x_2, x_3)$ is any solution of Eq. (16,17), then it is possible to produce a travelling wave in the following way:

$$\psi_v(t, x_1, x_2, x_3) = \psi(\gamma(t - vx_1), \gamma(x_1 - vt), x_2, x_3)$$

where $\gamma = 1/\sqrt{1 - v^2}$. $\psi_v$ is a travelling solution of NWE which moves in the $x_1$-direction with velocity $v = (v, 0, 0)$. 
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(18)

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Vieri Benci (DMA-Pisa)
Stationary waves
Travelling waves waves

Travelling waves waves
The following assumptions on $W$ guarantee the existence of solitons:

- (W-i) $W(s) \geq 0$;

- (W-ii) $W(0) = W'(0) = 0$; $W''(0) = 1$;

- (W-iii) $\exists s_0 : W(s_0) < \frac{1}{2} |s_0|^2$;
The following assumptions on $W$ guarantee the existence of solitons:

(W-i) $W(s) \geq 0$; this assumption can be weakened, but it is required in this form by physical models.

(W-ii) $W(0) = W'(0) = 0; \ W''(0) = 1$;

(W-iii) $\exists s_0 : W(s_0) < \frac{1}{2} |s_0|^2$;
The following assumptions on $W$ guarantee the existence of solitons:

- (W-i) $W(s) \geq 0$;

- (W-ii) $W(0) = W'(0) = 0; W''(0) = 1$; this is normalization condition; it would be sufficient to have $W''(0) > 0$.

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The following assumptions on $W$ guarantee the existence of solitons:

- **(W-i)** $W(s) \geq 0$;
- **(W-ii)** $W(0) = W'(0) = 0$; $W''(0) = 1$;
- **(W-iii)** $\exists s_0 : W(s_0) < \frac{1}{2} |s_0|^2$; this is the crucial assumption which characterize the potentials which might produce solitons.
By (W-ii) and (W-iii), we have that

\[ W(s) = \frac{1}{2}s^2 + N(s) \]

with

\[ N(s_0) < 0; \quad N(0) = N'(0) = 0 \]

Actually, \( N \) is the nonlinear term which, when it is negative, produces an attractive "force".
The main results of hylomorphic solitons are the following:

**Theorem**

If (W-i), (W-ii), (W-iii) hold then there exist orbitally stable standing waves of NWE having the form

\[ \psi(t, x) = u(|x|) e^{-i \omega t} \]

These solutions are the classical *Q-balls*. 
Main result for vortices in NWE

**Theorem**

*If (W-i),(W-ii),(W-iii) hold then, for every \( k \in \mathbb{Z} \), there exist solutions of NWE having the form*

\[
\psi(t,x) = u(r,x_3)e^{i(k\theta(x) - \omega t)}
\]

*where*

\[
x = (x_1, x_2, x_3)
\]

\[
r = \sqrt{x_1^2 + x_2^2}
\]

\[
\theta(x) = \text{Im} \left[ \log(x_1 + ix_2) \right]
\]

These solutions are ”rotating objects” which we might define as "protovorteces". In the physical literature there are results only in dimension 2.
Vortices with $k=3$ in NWE
Vortices with $k=8$ in NWE

NWE
Now let us consider the AGT

\[(\partial_t + iq\varphi)^2 \psi - (\nabla - iqA)^2 \psi + W'(\psi) = 0\]  \hspace{1cm} (19)

\[\nabla \cdot (\partial_t A + \nabla \varphi) = q \text{ Im} (\partial_t \psi \overline{\psi}) + q^2 \varphi |\psi|^2\]  \hspace{1cm} (20)

\[\nabla \times (\nabla \times A) + \partial_t (\partial_t A + \nabla \varphi) = q \text{ Im} (\nabla \psi \overline{\psi}) - q^2 A |\psi|^2\]  \hspace{1cm} (21)
Main result for Q-balls in AGT

Theorem

If (W-i), (W-ii), (W-iii) hold and the coupling constant $q$ is sufficiently small, then there exist solutions of eq. AGT having the form

$$
\psi(t, x) = u(|x|)e^{-i\omega t}
$$

$$
\varphi = \varphi(|x|)
$$

$$
A = 0
$$

These solutions are "spherical objects" surrounded by an electric field. No magnetic field is present.
Main result for vortices in AGT

**Theorem**

If (W-i),(W-ii),(W-iii) hold and the coupling constant $q$ is sufficiently small, then, for every $k \in \mathbb{Z}$, there exist solutions of eq. AGT having the form

\[
\psi(t, x) = u(r, x_3)e^{i(k\theta(x) - \omega t)}
\]

\[
\varphi = \varphi(r, x_3)
\]

\[
A = a(r, x_3)\vec{e}(r)
\]

where

\[
\vec{e}(r) = \frac{\nabla \theta(x)}{|\nabla \theta(x)|}
\]

These solutions are "three dimensional vortices". They produce a magnetic field which looks like the field created by a finite solenoid.
The two dimensional vortices in Gauge theories are very studied in physical literature.

However only the two dimensional case with double well potential is considered.

It is easy to prove that in dimension 3, the AGT with double well potential does not have vortices; in fact a necessary condition to have 3-d vortices is

$$W(0) = 0$$
Definition of matter density

Charge and energy allow to define a new quantity which is called *matter* (or Q-matter) whose density is given by

\[
\mu(t, x) = [\left| \rho_C(t, x) \right| - \rho_E(t, x)]^+
\]

where

\[
\rho_C(t, x) = \text{Im} \left( \partial_t \psi \bar{\psi} \right)
\]

is the charge density and

\[
\rho_E(t, x) = \frac{1}{2} \left| \partial_t \psi \right|^2 + \frac{1}{2} \left| \nabla \psi \right|^2 + W(\psi)
\]

is the energy density.
Proposition

If $|C| > \mathcal{E}$, then for all $t \in \mathbb{R}$ and some $x \in \mathbb{R}^3$,

$$\mu(t, x) \neq 0$$
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$$\mu(t, x) \neq 0$$

Proof.

Assume $C > 0$.

$$\int \mu(t, x) = \int [\rho_C(t, x) - \rho_E(t, x)]^+$$

$$\geq \int \rho_C(t, x) - \rho_E(t, x)$$

$$= C - \mathcal{E} > 0$$
Thus the "matter" does not disappear. Actually it has a tendency to coalesce in bumps and eventually to form solitons, namely very stable bumps.

This is the reason why we call this kind of solitons "hylomorphic": "hyle"="matter".
Existence of big bumps

If we make this extra assumptions on $W$:

$$\forall s \in (0, \delta_0), \ W(s) > \frac{1}{2}s^2 \ i.e. \ N(s) > 0$$

we have that:

**Proposition**

*Under the above assumption, if matter is present, there are "big" bumps, namely:* 

$$\mu(t,x) > 0 \Rightarrow |\psi(t,x)| > \delta_0$$
Proof of the proposition

We set

\[ \psi(t, x) = u(t, x)e^{iS(t, x)}; \quad k(t, x) = \nabla S(t, x); \quad \omega(t, x) = -\partial_t S(t, x) \]
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we have that

$$\rho_\mathcal{E}(t, x) = 2 \left| \partial_t u \right|^2 + 2 \left| \nabla u \right|^2 + \frac{1}{2} \left[ 1 + k^2 + \omega^2 \right] u^2 + N(u)$$

$$\rho_\mathcal{C}(t, x) = -\omega u^2$$
Proof of the proposition

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$$\psi(t, x) = u(t, x)e^{iS(t,x)}; \quad k(t, x) = \nabla S(t, x); \quad \omega(t, x) = -\partial_t S(t, x)$$

we have that

$$\rho_E(t, x) = \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} \left[ 1 + k^2 + \omega^2 \right] u^2 + N(u)$$

$$\rho_C(t, x) = -\omega u^2$$

thus,

$$\mu(t, x) = |\omega| u^2 - \frac{1}{2} |\partial_t u|^2 - \frac{1}{2} |\nabla u|^2 - \frac{1}{2} \left[ 1 + k^2 + \omega^2 \right] u^2 - N(u)$$

$$\leq \left[ |\omega| - \frac{1}{2} - \frac{1}{2} \omega^2 \right] u^2 - N(u)$$

$$\leq -N(u)$$
Proof of the proposition

Thus,

\[ \mu(t, x) > 0 \Rightarrow -N(u(t, x)) \geq \mu(t, x) > 0 \]

and by the fact that

\[ \forall s \in (0, \delta_0), \ N(s) > 0 \]  \hspace{1cm} (22)

we have that

\[ |\psi(t, x)| = u(t, x) > \delta_0 \]
If (22) does not hold, we might have bumps of matter arbitrarily small; in any case, working a little more, if matter is present, it is possible to show that

$$\min \lim_{t \to \infty} \| \psi(t, x) \|_{L_\infty} \geq \delta > 0$$

where $\delta$ depends only on $W$. 
Thus, given an initial condition in which,

\[ |C| > \mathcal{E}, \]

we have the presence of bumps of matter.
Thus, given an initial condition in which, 

$$|C| > \varepsilon,$$

we have the presence of bumps of matter.

As time goes on and two bumps of matter collide, several things might happen:

- they might bounce as two balls
- they might coalesce in a bigger bump
- they might have some kind of intermediate behaviour
Collision of solitons
Next, I will give an idea of the proves of the main existence results.

We will consider only the simplest case, namely the existence of spherical soliton in NWE.

Making the ansatz

$$\psi(t, x) = u(x) e^{-i\omega_0 t}$$

and replacing it in NWE, we get the elliptic equation:

$$-\Delta u + W'(u) = \omega_0^2 u$$  \hspace{1cm} (23)
Structure of the proves

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From what we have seen, it is clear that a the ratio

\[ \Lambda = \frac{\mathcal{E}}{\mathcal{C}} \]

plays a fundamental role. Actually, it is the base of our proves.
Structure of the proves

Given $u \in H^1$ and $\omega \in \mathbb{R}^+$, we set

$$\mathcal{E} (u, \omega) = \int \left[ \frac{1}{2} |\nabla u|^2 + W(u) dx \right] + \frac{1}{2} \omega^2 \int u^2 dx$$

$$\mathcal{C} (u, \omega) = \omega \int u^2 dx$$

If $\psi_0(t, x) = u_0(x)e^{-i\omega_0 t}$ is a standing wave, then $\mathcal{E} (u_0, \omega_0)$ is its energy and $\mathcal{C} (u_0, \omega_0)$ is its charge.
Structure of the proves

Given \( u \in H^1 \) and \( \omega \in \mathbb{R}^+ \), we set

\[
E(u, \omega) = \int \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] dx + \frac{1}{2} \omega^2 \int u^2 dx
\]

\[
C(u, \omega) = \omega \int u^2 dx
\]

If \( \psi_0(t, x) = u_0(x) e^{-i\omega_0 t} \) is a standing wave, then \( E(u_0, \omega_0) \) is its energy and \( C(u_0, \omega_0) \) is its charge.

In fact,

\[
E(\psi) = \int \left[ \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} \left( |k|^2 + \omega^2 \right) u^2 + W(u) \right] dx \quad (24)
\]

\[
C(\psi) = \int \omega u^2 dx \quad (25)
\]
A new variational principle

Theorem

\[ \psi_0(t,x) = u_0(x) e^{-i \omega_0 t} \text{ is a standing wave if and only if } (u_0, \omega_0) \text{ is a critical point (with } u \neq 0) \text{ of the functional } \mathcal{E}(u, \omega) \text{ constrained to the manifold} \]

\[ \mathcal{M}_\sigma = \{ (u, \omega) \in H^1 \times \mathbb{R} : C(u, \omega) = \sigma \} \]
Proof.

If \((u_0, \omega_0)\) is a critical point then

\[
\frac{\partial E}{\partial u}(u_0, \omega_0) = \lambda \frac{\partial C}{\partial u}(u_0, \omega_0)
\]

\[
\frac{\partial E}{\partial \omega}(u, \omega) = \lambda \frac{\partial C}{\partial \omega}(u_0, \omega_0)
\]

where \(\lambda\) is a lagrange multiplier. These equations written explicitly become

\[-\Delta u_0 + W'(u_0) + \omega_0^2 u = 2\lambda \omega_0 u_0\]

\[\omega_0 \int u^2 dx = \lambda \int u^2 dx\]

Since \(\int u^2 \neq 0\), \(\lambda = \omega_0\) and so, replacing this value in the first equation, we get eq. (23).
The above variational principle allows the following characterization of orbitally stable standing waves:

**Theorem**

If \((u_0, \omega_0)\) is a strict local minimum of the functional \(E(u, \omega)\) constrained to the manifold \(M_\sigma\), then \(\psi_0(t, x) = u_0(x) e^{-i\omega_0 t}\) is an orbitally stable standing wave.
The functional $\Lambda$

The ratio $\mathcal{E}/\mathcal{C}$, in this case, takes the form

$$\Lambda(u, \omega) = \frac{\int \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] dx}{\omega \int u^2 dx} + \frac{1}{2} \omega$$

If $(u, \omega) \in \mathcal{M}_\sigma$, then

$$\omega = \frac{\sigma}{\int u^2 dx}$$

so that $\mathcal{E}/\mathcal{C}$ takes the form

$$\Lambda_\sigma(u) := \Lambda(u, \omega(\sigma, u)) = \frac{1}{\sigma} \int \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] dx + \frac{\sigma}{2 \int u^2 dx}$$
Thus we get the following corollary:

**Corollary**

If $u_0(x)$ minimizes the functional

$$\Lambda_\sigma(u) = \frac{1}{\sigma} \int \left[ \frac{1}{2} |\nabla u|^2 + W(u) dx \right] + \frac{\sigma}{2} \int u^2 dx$$

then $\psi_0(t, x) = u_0(x)e^{-i\omega_0 t}$ is an orbitally stable standing wave with charge $\sigma$ and

$$\omega_0 = \frac{\sigma}{\int u_0^2 dx} \tag{26}$$
Idea of the proof

We consider the dynamical system associated to our equation. The phase space is given by $X = H^1(\mathbb{R}^N, \mathbb{C}) \times L^2(\mathbb{R}^N, \mathbb{C})$. To a function $\psi(t, \cdot) \in C^1(\mathbb{R}, H^1(\mathbb{R}, \mathbb{C}))$ we can associate a point

$$\Psi = (\psi, \psi_t) \in X.$$ 

Equation NWE can be written in the form

$$\frac{\partial \Psi}{\partial t} = A\Psi - W'(\Psi)$$

(27)

where $A$ is the matrix

$$A = \begin{pmatrix} 0 & 1 \\ \triangle & 0 \end{pmatrix}$$

and

$$W'(\Psi) = \begin{pmatrix} 0 \\ W'(\psi) \end{pmatrix}.$$
Idea of the proof

Let \((u_0, \omega_0)\) be a solution of (23); the set

\[
\Gamma_{\omega_0} = \left\{ \psi = (u_0(x - q)e^{i\theta}, -i\omega_0u_0(x - q)e^{i\theta}) : \psi \in H^1 \times L^2, \quad q \in \mathbb{R}^3, \quad \theta \in [0, 2\pi) \right\}
\]

is an invariant set. In fact the solution of (27) with initial conditions in \(\Gamma_{\omega_0}\) is given by

\[
\psi_{q,\theta}(t, x) = u_0(x - q)e^{-i(\omega_0t - \theta)}
\] (28)

Definition

The standing wave (28) is orbitally stable if and only if the set \(\Gamma_{\omega_0}\) is stable.
In order to prove that a set $\Gamma \subset X$ is stable, we shall use the following criterion:

**Proposition**

Assume that there exists a continuous function $V : X \rightarrow \mathbb{R}$ such that

- (i) $V(x) \geq 0$ and $V(x) = 0 \iff x \in \Gamma$
- (ii) $\frac{d}{dt} V(\Phi_t(x)) \leq 0$
- (iii) if $V(x_n) \rightarrow 0$, then $d(x_n, \Gamma) \rightarrow 0$

Then $\Gamma$ is stable.
In order to prove the theorem we shall use the above proposition with

$$V(\psi) = (\mathcal{E}(\psi) - m)^2 + (\mathcal{C}(\psi) - \sigma)^2$$

where $m$ is the minimum value of $\mathcal{E}(u, \omega)$ constrained to the manifold $\mathcal{M}_\sigma$.

(i) and (ii) are verified immediately. In order to prove (iii) a lot of hard analysis is needed. It is necessary to use concentration-compactness techniques as well as some new ideas.
Idea of the proof

It remains to see when the functional $\Lambda_\sigma(u)$ has a minimum.

Lemma

$$\inf_u \Lambda_\sigma(u) = \min_u \Lambda_\sigma(u)$$

if and only if

$$\inf_u \Lambda_\sigma(u) < 1$$

Actually, we proved that the functional $\Lambda_\sigma$ satisfies PS under the level 1. To do this it necessary to use both compactness and monotonicity methods.
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Idea of the proof

Lemma

\[ \inf_u \Lambda_{\sigma}(u) < 1 \]

if and only if

\[ \inf_s \frac{W(s)}{\frac{1}{2}s^2} := \inf_s \frac{\frac{1}{2}s^2 + N(s)}{\frac{1}{2}s^2} < 1 \]

if and only if

\[ N(s_0) < 0 \text{ for some } s_0 \in \mathbb{R} \]
Collision of solitons

Thank you for your attention!