Hardy inequalities and applications

Maria J. ESTEBAN

C.N.R.S. & Université Paris-Dauphine

In collaboration with: Adimurthi, Roberta Bosi, Jean Dolbeault, Michael Loss, Luis Vega
0 \leq \int_{\mathbb{R}^N} \left| \nabla u + \alpha \frac{x}{|x|^2} u \right|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + [\alpha^2 - (N - 2) \alpha] \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2},
Standard Hardy inequality

\[ 0 \leq \int_{\mathbb{R}^N} \left| \nabla u + \alpha \frac{x}{|x|^2} u \right|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + [\alpha^2 - (N - 2) \alpha] \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}, \]

Now we optimize over \( \alpha \) by choosing \( \alpha = (N - 2)/2 \), \( (\alpha^2 - (N - 2) \alpha) = -\frac{(N-2)^2}{4} \).
Standard Hardy inequality

$$0 \leq \int_{\mathbb{R}^N} \left| \nabla u + \alpha \frac{x}{|x|^2} u \right|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \left[ \alpha^2 - (N - 2) \alpha \right] \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2},$$

Now we optimize over $\alpha$ by choosing $\alpha = (N - 2)/2$, $(\alpha^2 - (N - 2) \alpha = -\frac{(N-2)^2}{4})$.

$$\int_{\mathbb{R}^N} |\nabla u|^2 \geq \frac{(N - 2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}, \quad \forall u,$$

and it is well known that $(N - 2)^2/4$ is optimal.
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and it is well known that \( (N - 2)^2/4 \) is optimal.

In operator terms, \( -\Delta - \frac{(N-2)^2}{4|x|^2} \geq 0 \) and

For all \( \nu > \frac{(N-2)^2}{4} \), \( -\Delta - \frac{\nu}{|x|^2} \) is not bounded below.
Local inequalities: \(0 \in \Omega, \quad \Omega \subset \mathbb{R}^n \text{ bounded}, \quad u = 0 \quad \text{on} \quad \partial \Omega\)

\[
\int_{\Omega} |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} \geq C_s \|u\|_{W^{s,2}}^2, \quad s \in (0,1). \tag{1}
\]
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\]

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\int_{\Omega} |\nabla u|^p - \left( \frac{n-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} \geq 0, \tag{3}
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Local inequalities: \( 0 \in \Omega, \ \Omega \subset \mathbb{R}^n \) bounded, \( u = 0 \) on \( \partial \Omega \)

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\] (3)

Let \( 1 < p \leq n \). Let \( R \) be large enough. There exists a constant \( C > 0 \) depending on \( n, p \) and \( R > 0 \) such that

\[
\int_{\Omega} |\nabla u|^p - \left( \frac{n-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} \geq C \int_{\Omega} \frac{|u|^p}{|x|^p (\log R/|x|)^\gamma} ,
\] (3)

(i) \( \gamma \geq 2 \) when \( 1 < p < n \), \( C = \frac{1}{4} \) if \( p = 2 \) and \( \gamma = 2 \)

(ii) \( \gamma \geq n \) when \( p = n \), \( C = \left( \frac{n-1}{n} \right)^n \)
Let $k \geq 1$ be an integer and $1 < p \leq n$. Then, if $R > e^{(k-1) \sup_{x \in \Omega} |x|}$, there exist constants $C = C(p, n) > 0$ and $\lambda = \lambda(\Omega, p, R) < 0$ such that for all $u \in W^{1,p}(\Omega)$,

$$
\int_{\Omega} |\nabla u|^p - \left( \frac{n - p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} \geq \left( \sum_{j=1}^{k} \frac{1}{\left( \log(j) \frac{R}{|x|} \right)^{2}} \right) \frac{|u|^p}{|x|^p} + \lambda \int_{\partial\Omega} |u|^p, \quad (p < n)
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\[
\int_{\Omega} |\nabla u|^p = \left( \frac{n - p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} \geq \frac{C}{n} \int_{\Omega} \left( \sum_{j=2}^{k} \frac{1}{(\log(j) R/|x|)^n} \right) \frac{|u|^n}{|x|^n} + \lambda \int_{\partial \Omega} |u|^p, \quad (p < n) \tag{5}
\]

\[
\int_{\Omega} |\nabla u|^n = \left( \frac{n - 1}{n} \right) \int_{\Omega} \frac{|u|^n}{(|x| \log \frac{R}{|x|})^n} \geq \frac{C}{n} \int_{\Omega} \left( \sum_{j=2}^{k} \frac{1}{(\log(j) R/|x|)^n} \right) \frac{|u|^n}{|x|^n} + \lambda \int_{\partial \Omega} |u|^n, \quad (p = n) \tag{4}
\]
\[
0 \leq \int_{\mathbb{R}^N} \left| \nabla u + h(x) \frac{x}{|x|^2} u \right|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} \left[ \frac{h^2}{r^2} - \text{div} \left( \frac{hx}{|x|^2} \right) \right] |u|^2 \, dx,
\]
Optimization of the controlled potential: (Dolbeault + E. + Loss + Vega)

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If \( h(x) = h(|x|) \), then

\[ V_\infty := \text{div} \left( \frac{h x}{|x|^2} \right) - \frac{h^2}{r^2} = \frac{1}{r^2} \left( (n - 2) h + h' r - h^2 \right) . \]
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\]

If \( h = \frac{n-2}{2} + \ell \),

\[
(n-2)h + h'r - h^2 = \frac{(n-2)^2}{4} + \ell'r - \ell^2.
\]
Theorem 1  Let $A$ denote the class of the functions $n$, continuous in the interval $[0, \delta)$ for some $\delta > 0$, and such that $n(0) = 0$. Then, for all $k \geq 1$, 

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\sup_{n \in A} \left\{ \liminf_{s \to 0^+} \left( s \ell'(s) - \ell^2(s) - \frac{1}{4} \sum_{j=1}^{k-1} X_1^2(s) \cdots X_j^2(s) \right) X_1^{-2}(s) \cdots X_k^{-2}(s) \right\} = \frac{1}{4}.
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where $X_1(s) := (a - \log(s))^{-1}$ for some $a > 1$, $X_k(s) := X_1 \circ X_{k-1}$. The functions $X_k$ are well defined for $|x| = s < e^{a-1}$.
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Idea of proof: 

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\ell(s) = \frac{1}{2} X_1(s) \left( 1 - 2 \ell_1 \left( \frac{1}{X_1(s)} \right) \right).
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V_\infty = \frac{1}{r^2} \left( (n - 2) h + h' r - h^2 \right) = \frac{1}{r^2} \left[ \frac{(n - 2)^2}{4} + \frac{1}{4 |\log r|^2} \left[ 1 + \left[ \frac{1}{|\log \log r|^2} + \cdots \right] \right] \right].
$$
Consider \( V(x) = \frac{(N-2)^2}{8} \left( \frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) \), \( |y| = d > 0 \).
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$$\int_{\mathbb{R}^N} \left| \nabla u \right|^2 - \frac{(N-2)^2}{8} \int_{\mathbb{R}^N} \left( \frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) |u|^2$$

$$= \frac{1}{2} \sum_{\pm} \int_{\mathbb{R}^N} \left| \nabla u_{\pm} \right|^2 - \frac{(N-2)^2}{4} \frac{|u_{\pm}|^2}{|x|^2} \geq 0$$

where $u_{\pm}(\cdot) = u(\cdot \pm y)$. 

**Multipolar potentials. 2 singularities.**
Consider $V(x) = \frac{(N-2)^2}{8} \left( \frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right)$, $|y| = d > 0$.

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If $V(x) = \nu \left( \frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right)$, $0 < \nu \leq \frac{(N-2)^2}{8}$

can we get a better estimate?
Multipolar potentials. 2 singularities.

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= \frac{1}{2} \sum_{\pm} \int_{\mathbb{R}^N} |\nabla u_\pm|^2 - \frac{(N-2)^2}{4} \frac{|u_\pm|^2}{|x|^2} \geq 0
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can we get a better estimate?

What happens if \( \nu > \frac{(N-2)^2}{8} \) ?
Recently T & M Hoffmann-Ostenhof, A. Laptev and J. Tidblom have studied Hardy-like inequalities for multiparticle systems:

\[
\sum_{j=1}^{P} \int_{\mathbb{R}^P} \left| \nabla_{x_j} u \right|^2 \, dx \geq C(N, P) \sum_{1 \leq i \leq j \leq P} \int_{\mathbb{R}^P} \frac{|u|^2}{|x_i - x_j|^2} \, dx
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Recently T & M Hoffmann-Ostenhof, A. Laptev and J. Tidblom have studied Hardy-like inequalities for multiparticle systems:

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\sum_{j=1}^{P} \int_{\mathbb{R}^PN} |\nabla_{x_j} u|^2 \, dx \geq C(N, P) \sum_{1 \leq i < j \leq P} \int_{\mathbb{R}^PN} \frac{|u|^2}{|x_i - x_j|^2} \, dx
\]

Felli, Marchini, Terracini have recently been involved in the analysis of when the operator

\[
-\Delta - \sum_{i=1}^{M} \frac{c_i}{|x - y_i|^2} > 0
\]
Spectral meaning of the inequalities

Our viewpoint is different, we do not want to find for which $c_i$’s

\[-\Delta - \sum_{i=1}^{M} \frac{c_i}{|x - y_i|^2} \geq 0,\]
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$$-\Delta - \sum_{i=1}^{M} \frac{c_i}{|x - y_i|^2} \geq 0,$$

but given the $c_i$’s, we want to find lower bounds:

$$-\Delta - \sum_{i=1}^{M} \frac{c_i}{|x - y_i|^2} \geq -L > -\infty,$$
Spectral meaning of the inequalities

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\]

\( L \) depending on the \( c_i \)'s and on the spatial distribution of the \( y_i \)'s or equivalently,

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + L \int_{\mathbb{R}^N} |u|^2 \, dx \geq \sum_{i=1}^{M} \int_{\mathbb{R}^N} \frac{c_i |u|^2}{|x - y_i|^2} \, dx , \quad \forall u ,
\]

which means that

\[
\lambda_1 \left( -\Delta - \sum_{i=1}^{M} \frac{c_i}{|x - y_i|^2} \right) \geq -L .
\]
Why is this issue interesting?

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3) In general, in solid state physics problems, band gaps, etc.

But of course, only as an intermediate technical tool, since the problem we look at corresponds to only 1 electron.
Remark.- Since $\Delta$ and $1/|x|^2$ have the same scaling properties,

$$
\lambda_1 \left( -\Delta - \sum_{i=1}^{M} \frac{c_i}{|x - y_i|^2} \right) = \frac{1}{\ell^2} \lambda_1 \left( -\Delta - \sum_{i=1}^{M} \frac{c_i}{|x - \frac{y_i}{\ell}|^2} \right)
$$
Remark. Since $\Delta$ and $1/|x|^2$ have the same scaling properties,

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For instance, in the case $M = 2$, if $|y| = d$,

$$\lambda_1 \left( -\Delta - \left( \frac{1}{|x - y|^2} + \frac{1}{|x + y|^2} \right) \right) = \frac{1}{d^2} \lambda_1 \left( -\Delta - \left( \frac{1}{|x - y_1|^2} + \frac{1}{|x + y_1|^2} \right) \right) = \frac{C}{d^2}$$

with $|y_1| = \frac{|y|}{d} = 1$. 
Simple case \( c_1 = \cdots = c_M = \nu \in (0, (N - 2)^2/4) \) or \( \nu = (N - 2)^2/4 \).

Very well known (truncation) method : IMS (Ismagilov, Morgan-Simon, Sigal)

If \( M \) singularities, let \( (J_k)_{k=1}^{M+1} \) be a \( W^{1,\infty}(\mathbb{R}^N) \) partition of unity in the sense that
\[
\sum_{k=1}^{M+1} J_k^2 = 1 \quad \text{and} \quad \text{int} \ (\text{supp} J_i) \cap \text{int} \ (\text{supp} J_j) = \emptyset \quad \text{if} \ i \neq j.
\]

For \( k = 1, \ldots, M, \ J_k \equiv 1 \) in a ball centered at \( y_k \) and supported in a larger ball also centered at \( y_k \).
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\int_{\mathbb{R}^N} (|\nabla u|^2 - V|u|^2) = \sum_{k=1}^{M+1} \int_{\mathbb{R}^N} (|\nabla (J_k u)|^2 - V|(J_k u)|^2) - \int_{\mathbb{R}^N} \left( \sum_{k=1}^{M+1} |\nabla J_k|^2 \right) |u|^2.
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If for \( k = 1, \ldots, M \), on \( \text{supp} J_k \), \( V(x) \leq \frac{\nu}{|x - y_k|^2} + L_k \).
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For $k = 1, \ldots, M$, $J_k \equiv 1$ in a ball centered at $y_k$ and supported in a larger ball also centered at $y_k$.

$$\int_{\mathbb{R}^N} (|\nabla u|^2 - V \, |u|^2) = \sum_{k=1}^{M+1} \int_{\mathbb{R}^N} (|\nabla (J_k u)|^2 - V \, |(J_k u)|^2) - \int_{\mathbb{R}^N} \left( \sum_{k=1}^{M+1} |\nabla J_k|^2 \right) |u|^2.$$

If for $k = 1, \ldots, M$, on $\text{supp} \, J_k$, $V(x) \leq \frac{\nu}{|x-y_k|^2} + L_k$.

and if $c_k \leq \frac{(N-2)^2}{4}$ and $1 \leq k \leq M$,

$$\int_{\mathbb{R}^N} (|\nabla (J_k u)|^2 - V \, |(J_k u)|^2) \geq -L_k \int_{\mathbb{R}^N} |J_k \, u|^2 \, dx$$
\[
\int_{\mathbb{R}^N} (|\nabla (J_{M+1} u)|^2 - V |(J_{M+1} u)|^2) \geq - \sup (V J_{M+1}^2) \int_{\mathbb{R}^N} |u|^2 \, dx.
\]

If \( d = \min_{i \neq j} |y_i - y_j| \), we can choose the \( J_k \)'s such that

\[
\sum_{k=1}^{M+1} |\nabla J_k|^2 \leq \frac{\pi^2}{d^2}.
\]
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\int_{\mathbb{R}^N} \left( |\nabla (J_{M+1} u)|^2 - V |(J_{M+1} u)|^2 \right) \geq - \sup (V J_{M+1}^2) \int_{\mathbb{R}^N} |u|^2 \, dx.
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Finally, controlling the remainder a little bit more carefully than above, one obtains

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \left( \frac{M \nu}{d^2} + \frac{\pi^2}{d^2} \right) \int_{\mathbb{R}^N} |u|^2 \, dx \geq \sum_{i=1}^{M} \int_{\mathbb{R}^N} \frac{\nu |u|^2}{|x - y_i|^2} \, dx \geq 0, \quad \forall u,
\]
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\]

\[
\ldots \quad \text{and that implies that}
\]

\[
\lambda_1 \left( - \Delta - \sum_{i=1}^{M} \frac{\nu}{|x - y_i|^2} \right) \geq - \frac{M \nu}{d^2} - \frac{\pi^2}{d^2} .
\]
At least for $M = 2$ we know how to use a more refined IMS method (new idea) to pass from

$$\lambda_1 \left( -\Delta - \sum_{i=1}^{2} \frac{\nu}{|x - y_i|^2} \right) \geq -\frac{2\nu}{d^2} - \frac{\pi^2}{d^2}.$$ 

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But anyway, these asymptotics are bad for $d$ small! Indeed, for instance, if $\nu \leq \frac{(N-2)^2}{4M}$, $\lambda_1 \geq 0$, for all $d > 0$. 
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But anyway, these asymptotics are bad for $d$ small! Indeed, for instance, if $\nu \leq \frac{(N-2)^2}{4M}$, $\lambda_1 \geq 0$, for all $d > 0$.

Let us go back to the idea that Hardy inequalities can be obtained by expanding squares.
Easy case: 2 singularities at $\pm y$, $c_1 = c_2 = \nu$

\[
0 \leq \int_{\mathbb{R}^N} \left| \nabla u + \alpha \frac{x - y}{|x - y|^2} u + \alpha \frac{x + y}{|x + y|^2} u \right|^2 \\
= \int_{\mathbb{R}^N} |\nabla u|^2 + \left[ \alpha^2 - (N - 2) \alpha \right] \int_{\mathbb{R}^N} |u|^2 \left( \frac{1}{|x - y|^2} + \frac{1}{|x + y|^2} \right) \\
+ 2 \alpha^2 \int_{\mathbb{R}^N} |u|^2 \frac{(x - y) \cdot (x + y)}{|x - y|^2 |x + y|^2}
\]
Easy case: 2 singularities at $\pm y$, $c_1 = c_2 = \nu$

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$$= \int_{\mathbb{R}^N} |\nabla u|^2 + \left[ \alpha^2 - (N-2)\alpha \right] \int_{\mathbb{R}^N} |u|^2 \left( \frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right)$$

$$+ 2 \alpha^2 \int_{\mathbb{R}^N} |u|^2 \frac{(x-y) \cdot (x+y)}{|x-y|^2 |x+y|^2}$$

and now we can estimate the last term of the above expression and optimize in $\alpha$. 
Easy case: 2 singularities at $\pm y$, $c_1 = c_2 = \nu$

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0 \leq \int_{\mathbb{R}^N} \left| \nabla u + \alpha \frac{x-y}{|x-y|^2} u + \alpha \frac{x+y}{|x+y|^2} u \right|^2
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\[
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\]

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+ 2 \alpha^2 \int_{\mathbb{R}^N} |u|^2 \frac{(x-y) \cdot (x+y)}{|x-y|^2 |x+y|^2}
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and now we can estimate the last term of the above expression and optimize in $\alpha$.

\[
(*) \quad 2 \frac{(x-y) \cdot (x+y)}{|x-y|^2 |x+y|^2} \leq \beta \left( \frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) + \gamma.
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**Lemma.** The best $\gamma$ in (*) is equal to $\frac{(1-\beta)^2}{4 \beta d^2}$.

\[
\int_{\mathbb{R}^N} |\nabla u|^2 - [(N-2) \alpha - \alpha^2 (1+\beta)] \int_{\mathbb{R}^N} |u|^2 \left( \frac{1}{|x-y|^2} + \frac{1}{|x+y|^2} \right) \geq - \frac{(1-\beta)^2}{4 \beta d^2} \alpha^2 \int_{\mathbb{R}^N} |u|^2.
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\]

and now, $(N-2) \alpha - \alpha^2 (1+\beta) = \nu$ and optimize in $\alpha, \beta$ to minimize $\frac{(1-\beta)^2}{4 \beta d^2} \alpha^2$. 
By taking the optimal $\alpha, \beta$, we find an explicit constant $L(\nu, N)$:

$$\lambda_1\left(-\Delta - \left(\frac{\nu}{|x-y|^2} + \frac{\nu}{|x+y|^2}\right)\right) \geq -\frac{L(\nu, N)}{d^2},$$

and as corollaries of the concrete expression for $L(\nu, N)$, we have:

1. The map $L(\nu, N)$ is nondecreasing.
2. If $\left(\frac{N}{2}\right)^2 < \frac{8}{\nu}; L(\nu, N) = 0$ for all $d > 0$.
3. If $\left(\frac{N}{2}\right)^2 < \frac{4}{\nu}; 0 < L(\nu, N) < 1$ for all $d > 0$.
4. $L(\nu, N) = +1$ for all $\nu > \left(\frac{N}{2}\right)^2$.
By taking the optimal $\alpha$, $\beta$, we find an explicit constant $L(\nu, N)$:

$$\lambda_1 \left( -\Delta - \left( \frac{\nu}{|x-y|^2} + \frac{\nu}{|x+y|^2} \right) \right) \geq -\frac{L(\nu, N)}{d^2},$$

and as corollaries of the concrete expression for $L(\nu, N)$, we have:

- The map $\nu \mapsto L(\nu, N)$ is nondecreasing.
- If $\nu \leq \frac{(N-2)^2}{8}$, \quad $L(\nu, N) = 0$ for all $d > 0$.
- If $\frac{(N-2)^2}{8} < \nu \leq \frac{(N-2)^2}{4}$, \quad $0 < L(\nu, N) < +\infty$ for all $d > 0$.
- $L(\nu, N) = +\infty$ for all $\nu > \frac{(N-2)^2}{4}$. 

Remark. This estimate can never be optimal for the computation of the first eigenvalue because above we have estimated some potential terms pointwise, and the eigenfunctions are not Dirac masses!

Remark. We can do the same (more complicated, but same ideas) for more than 2 singularities.
By taking the optimal $\alpha$, $\beta$, we find an explicit constant $L(\nu, N)$:

$$
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IMS, which is the method traditionally used in this kind of business, is good for distant singularities, but the square expanding method gives much better estimates for nearby singularities.
Comparison between the two methods

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In a recent work Javier Duoandikoetxea and Luis Vega have explored a new (geometrical) method to compute the lowest eigenvalues of $-\Delta - V$ in $\mathbb{R}^N$ for a nonnegative potential $V$ in $M^{N/2}(\mathbb{R}^N)$ with only one singularity.

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Right now we are working on the extension of this new method to the case of several singularities, and we are optimistic about getting much better estimates for $d$ small than with the other 2 methods. To follow...
Same kind of things can be done for other operators... like the Dirac operator (first order and unbounded from below!).
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THE END