Homework No. 1: Rudin’s book, page 138, problems 1, 2, 3, 5, 6 and 7. Prove that if $f(x) = 0$ for $x \in [0,1] - \mathbb{Q}$ and $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$ with $m,n$ relatively prime, then $f$ is Riemann integrable. Due Wednesday, September 7, 2011 by 5:00pm.

Homework No. 2: Rudin’s book, pages 138-140, problems 8, 10, 11, 15, and 16. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $f,g \in \mathcal{R}(\alpha)$. Prove that $h(x) = \phi(f(x),g(x))$ defines a function in $\mathcal{R}(\alpha)$. Due Wednesday, September 21, 2011 by 5:00pm.

Homework No. 3:
1. Prove that $\{(x,y); 1 \leq x, y \leq 2, x \text{ is rational}\}$ is not an elementary set.
2. Prove that the outer measure of the Cantor ternary set is 0.
3. Using only the definition of $m^*$, prove that if $m^*(A_n) = 0$ for $n = 1, 2, \ldots$ then $m^*(\bigcup_{n=1}^{\infty} A_n) = 0$.
4. Prove that if $f_n : [0,1] \to \mathbb{R}$ is a sequence of Riemann integrable functions then $m^*\{(x,f_n(x)); x \in [0,1], n = 1, 2, \ldots\} = 0$. Due Wednesday, September 28, 2011 by 5:00pm.

Homework No. 4
1. Let $D \subset \mathbb{R}$ be dense, $\Omega \subset \mathbb{R}^n$ be measurable, and $f : \Omega \to \mathbb{R}$. Prove that if $\{x; f(x) > \alpha\}$ is measurable for all $\alpha \in D$ then $f$ is measurable.
2. Let $E \subset \mathbb{R}$ and $x \in \mathbb{R}$. Prove that $m^*(E) = m^*\{x + t; t \in E\}$ (The outer measure is invariant under translations).
3. For $x, y \in [0,1]$ define $x \sim y$ if $x - y$ is a rational number. Prove that $\sim$ defines an equivalence relation. For each equivalence class choose a representative and call $C$ the set of all such representatives. For each rational number $y$ let $C_y = \{t \in [0,1]; t = y + x, \text{ or } t = y + x - 1 \text{ for some } x \in C\}$. Prove that $\bigcup_y \text{ rational in } [0,1] C_y = [0,1]$. Prove that $m^*(C_y)$ is independent of $y$. Conclude that $C$ cannot be measurable. (Hint: Either $m^*(C_y) = 0$ or $m^*(C_y) > 0$.)
4. Let $D \subset \mathbb{R}^n$ be measurable. We say that $u \in L^\infty(D)$ if $u : D \to \mathbb{R} \cup \{-\infty, +\infty\}$ is measurable and $m(\{x; |u(x)| > M\}) = 0$, for some $M \in [0,\infty)$. Define $\|u\|_\infty = \inf\{M; m(\{x; |u(x)| > M\}) = 0\}$. Prove that $\|u + v\|_\infty \leq \|u\|_\infty + \|v\|_\infty$.
5. Prove that $L^\infty(D)$ is complete under the metric given by $d(u,v) = \|u - v\|_\infty$. 

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Due Wednesday, October 5, 2011 by 5:00pm.

Homework No. 5

Solve problems 2, 3, 4, and 5 in pages 332 - 333 of Rudin’s book.

For \( x \in (0, 1) \) let \( f(x) = x^{-1/2} \). Use the monotone convergence theorem to prove that \( f \in L^1(0, 1) \) but \( f \not\in L^2(0, 1) \). Feel free to use that the series \( \sum_{n=1}^{\infty} n^{-a} \) converges if and only if \( n > 1 \).

Due October 12, 2011 by 5:00pm

Homework No. 6

Solve problems 10, 12, 13, and 15 in Chapter 11 of Rudin’s book.

Prove that if \( A \) is a measurable subset of \((0, +\infty)\) and \( c > 0 \) then \( m\{ac; a \in A\} = cm(A) \).

Homework No. 7

1. Let \( f : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^m \) defined by \( f(x) = (f_1(x), \ldots, f_m(x)) \). Prove that if \( f \) is differentiable at \( x_0 \) then each \( f_i, i = 1, \ldots, m, \) is differentiable at \( x_0 \).

2. Let \( H_1 = \{ u \in L^2([0, 2\pi]); u' \in L^2([0, 2\pi]) \} \). Prove that \( H_1 \) is a Hilbert space under the inner product \( <u, v> = \int_0^{2\pi} (u(x)v(x) + u'(x)v'(x))\,dx \)


4. Prove that a Banach space (complete normed vector space) is a Hilbert space (its norm is defined by an inner product) if and only if its norm satisfies \( \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \).

5. Prove that \( C([0, 1], \mathbb{R}) \) is not a Hilbert space under the norm \( \|u\| = \max\{|u(x)|; x \in [0, 1]\} \).

Due November 16, 2011 by 5:00pm.

Homework No. 8

Solve problems 13, 14, 15, 16, 17, and 19 (pages 240-241) in Rudin’s book.

Due Wednesday, November 30, 2011 by 5:00pm.

Homework No. 9

1. Let \( \theta : \mathbb{R}^N \to \mathbb{R}^N \) be an orthogonal transformation \(<\theta(x), \theta(y)> = <x, y>\). Prove that \( m^*(A) = m^*(\theta(A)) \) for any \( A \subset \mathbb{R}^N \).
2. Let $H$ be a linear subspace of $\mathbb{R}^N$ with $\text{dim}(H) < n$. Prove that $m(H) = 0$.

3. Prove that the measure of the set of boundary points of a parallelipiped in $\mathbb{R}^N$ is zero. Hint: If $K = P(v_1, \ldots, v_N)$, consider $\{c_1 v_1 + \cdots + c_n v_N; c_i \in \{0, 1\} \text{ for some } i \}$.

4. Let $K = P(v_1, \ldots, v_N)$. Prove there exist $c > 0$, independent of $j$, such that

$$m\{x \in \mathbb{R}^N; d(x, \partial K) < 2/j\} \leq c/j.$$

5. Let $K$ be as in problem 4 and $(p, h) \in \mathbb{R}^{N+1}$ with $p \in \mathbb{R}^N, h \in [0, \infty)$. For a positive integer, let $A_1 = \{(x, 0) + s(p, h); x \in K, s \in [0, 1/j]\}$, and $A_2 = x \in K; x + (1/j)p \in K$.

Prove that

$$m(K) - \frac{c}{j} \cdot \frac{1}{j} \leq m(A_1) \leq m(K) + \frac{c}{j} \cdot \frac{1}{j}.$$ 

Hint: Prove that $A_2 \times [0, 1/j] \subset A_1 \subset \{x + sp; x \in K, s \in [0, 1/j]\}$.

6. Let $B_i = \{(i/j)(p, h) + z; z \in A_1\}, i = 1, \ldots, j - 1$. Prove that $m(B_i) = m(A_1)$.

7. Using mathematical induction prove that $m(P(\varphi_1, \ldots, \varphi_n)) = |\text{det}(\varphi_1, \ldots, \varphi_n)|$.

Due Wednesday, December 7, 2011 by 5:00pm.