MATH 132: Mathematical Analysis II
Fall 2011
Professor Alfonso Castro

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Homework No. 1: Rudin's book, page 138, problems 1, 2, 3, 5, 6 and 7. Prove that if $f(x) = 0$ for $x \in [0,1] - \mathbb{Q}$ and $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$ with m, n reatively prime, then f is Riemann integrable. Due Wednesday, September 7, 2011 by 5:00pm.

Homework No. 2: Rudin's book, pages 138-140, problems 8, 10, 11, 15, and 16. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $f, g \in \mathcal{R}(\alpha)$. Prove that $h(x) = \phi(f(x), g(x))$ defines a function in $\mathcal{R}(\alpha)$. Due Wednesday, September 21, 2011 by 5:00pm.

Homework No. 3:

1. Prove that $\{(x,y); 1 \le x, y \le 2, x$ is rational $\}$ is not an elementary set.

2. Prove that the outer measure of the Cantor terniary set is 0.

3. Using only the definition of m^*, prove that if $m^*(A_n) = 0$ for $n = 1, 2, \ldots$ then $m^*(\cup_{n=1}^{\infty} A_n) = 0$.

4. Prove that if $f_n : [0,1] \to \mathbb{R}$ is a sequence of Riemann integrable functions then $m^*\{(x, f_n(x)); x \in [0,1], n = 1, 2, \ldots\} = 0$.

Due Wednesday, September 28, 2010 by 5:00pm.

Homework No. 4

1. Let $D \subset \mathbb{R}$ be dense, $\Omega \subset \mathbb{R}^n$ be measurable, and $f : \Omega \to \mathbb{R}$. Prove that if $\{x; f(x) > \alpha\}$ is measurable for all $\alpha \in D$ then f is measurable.

2. Let $E \subset \mathbb{R}$ and $x \in \mathbb{R}$. Prove that $m^*(E) = m^*\{x + t; t \in E\}$ (The outer measure is invariant under translations).

3. For $x, y \in [0,1]$ define $x\,\tilde{}\,y$ if $x - y$ is a rational number. Prove that $\tilde{}$ defines an equivalence relation. For each equivalence class choose a representative and call C the set of all such representatives. For each rational number y let $C_y = \{t \in [0,1]; t = y + x,$ or $t = y + x - 1$ for some $x \in C\}$. Prove that $\cup_{y \text{ rational in } [0,1]} C_y = [0,1]$. Prove that $m^*(C_y)$ is independent of y. Conclude that C cannot be mesaurable. (Hint: Either $m^*(C_y) = 0$ or $m^*(C_y) > 0$.)

4. Let $D \subset \mathbb{R}^n$ be measurable. We say that $u \in L^\infty(D)$ if $u : D \to \mathbb{R} \cup \{-\infty, +\infty\}$ is measurable and $m(\{x; |u(x)| > M\}) = 0$, for some $M \in [0, \infty)$. Define $\|u\|_\infty = \inf\{M; m(\{x; |u(x)| > M\}) = 0\}$. Prove that $\|u + v\|_\infty \le \|u\|_\infty + \|v\|_\infty$.

5. Prove that $L^\infty(D)$ is complete under the metric given by $d(u,v) = \|u - v\|_\infty$.

Due Wednesday, October 5, 2011 by 5:00pm.

Homework No. 5

Solve problems 2, 3, 4, and 5 in pages 332 - 333 of Rudin's book.

For $x \in (0,1)$ let $f(x) = x^{-1/2}$. Use the monotone convergence theorem to prove that $f \in L^1(0,1)$ but $f \notin L^2(0,1)$. Feel free to use that the series $\sum_{n=1}^{\infty} n^{-a}$ converges if and only if $n > 1$.

Due October 12, 2011 by 5:00pm

Homework No. 6

Solve problems 10, 12, 13, and 15 in Chapter 11 of Rudin's book.

Prove that if A is a measurable subset of $(0,+\infty)$ and $c > 0$ then $m\{ac; a \in A\} = cm(A)$.

Homework No. 7

1. Let $f : \Omega(\subset \mathbb{R}^n) \to \mathbb{R}^m$ defined by $f(x) = (f_1(x), \ldots, f_m(x))$. Prove that if f is differentiable at x_0 then each f_i, $i = 1, \ldots, m$, is differentiable at x_0.

2. Let $H_1 = \{u \in L^2([0,2\pi]); u' \in L^2([0,2\pi])\}$. Prove that H_1 is a Hilbert space under the inner product $< u,v >= \int_0^{2\pi}(u(x)v(x) + u'(x)v'(x))dx$

3. Solve problem 16 in page 334 of Rudin's book.

4. Prove that a Banach space (complete normed vector space) is a Hilbert space (its norm is defined by an inner product) if and only if its norm satisfies $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

5. Prove that $C([0,1],\mathbb{R})$ is not a Hilbert space under the norm

$$\|u\| = \max\{|u(x)|; x \in [0,1]\}.$$

Due November 16, 2011 by 5:00pm.

Homework No. 8

Solve problems 13, 14, 15, 16, 17, and 19 (pages 240-241) in Rudin's book.

Due Wednesday, November 30, 2011 by 5:00pm.

Homework No. 9

1. Let $\theta : \mathbb{R}^N \to \mathbb{R}^N$ be an orthogonal transformation ($< \theta(x), \theta(y) >=< x, y >$). Prove that $m^*(A) = m^*(\theta(A))$ for any $A \subset \mathbb{R}^N$.

2. Let H be a linear subspace of $\mathbb{R}^N$ with $dim(H) < n$. Prove that $m(H) = 0$.

3. Prove that the measure of the set of boundary points of a parallelipiped in $\mathbb{R}^N$ is zero. Hint: If $K = P(v_1, \ldots, v_N)$, consider $\{c_1v_1 + \cdots + c_nv_N; c_i \in \{0, 1\}$ for some i $\}$.

4. Let $K = P(v_1, \ldots, v_N)$, Prove there exist $c > 0$, independent of j, such that $m\{x \in \mathbb{R}^N; d(x, \partial K) < 2/j\} \leq c/j$.

5. Let K be as in problem 4 and $(p, h) \in \mathbb{R}^{N+1}$ with $p \in \mathbb{R}^N, h \in [0, \infty)$. For j positive integer, let $A_1 = \{(x, 0) + s(p, h); x \in K, s \in [0, 1/j]$, and $A_2 = x \in K; x + (1/j)p \in K\}$ Prove that

$$(m(K) - \frac{c}{j})\frac{1}{j} \leq m(A_1) \leq (m(K) + \frac{c}{j})\frac{1}{j}.$$

Hint: Prove that $A_2 \times [0, 1/j] \subset A_1 \subset \{x + sp; x \in K, s \in [0, 1/j]\}$.

6. Let $B_i = \{(i/j)(p, h) + z; z \in A_1\}$, $i = 1, \ldots, j - 1$. Prove that $m(B_i) = m(A_1)$.

7. Using mathematical induction prove that $m(P(\varphi_1, \ldots, \varphi_n)) = |det(\varphi_1, \ldots, \varphi_n|$.

Due Wednesday, December 7, 2011 by 5:00pm.