MATH 182: PARTIAL DIFFERENTIAL EQUATIONS
Fall 2008
Professor Alfonso Castro

Homework No. 1
1. Prove that
   \[S(A_1 \cup A_2, B_1 \cup B_2), S(A_1 \cap A_2, B_1 \cap B_2), \text{ and } S(A_1 - A_2, B_1 - B_2)\]
   are subsets of \(S(A_1, B_1) \cup S(A_2, B_2)\).
2. Prove that if \(A, B\) are finitely measurable so are \(A \cup B, A \cap B, \text{ and } A - B\).
3. Prove that if \(A\) is countable then \(m^*(A) = 0\).
4. Prove that if \(f\) and \(g\) are measurable functions so are \(-g, f + g\) and \(f \cdot g\).
5. Let \(X \subset \mathbb{R}^n\), and \(f : X \to [-\infty, \infty]\) be measurable. Prove that \(\{x \in X; f(x) = +\infty\}\) is measurable.
6. Let \(X \subset \mathbb{R}^n\) be measurable. Prove that if \(f : \mathbb{R} \to \mathbb{R}\) is continuous and \(g : X \to \mathbb{R}\) is measurable then \(f \circ g : X \to \mathbb{R}\) is measurable.
7. Prove that if \(\{f_k\}_k\) is a sequence of measurable functions then \(\lim \inf f_n\) and \(\lim \sup f_n\) are measurable.

Due Tuesday, February 3, 2009 by 5:00pm.

Homework No. 2
1. State and prove Fatou’s lemma.
2. State and prove the monotone convergence theorem.
3. State and prove the dominated convergence theorem.
4. Prove that the set of trigonometric polynomials is dense in \(L^2(0, 2\pi)\) (A trigonometric polynomial is a function of the form \(f(x) = a_1 + \cdots + a_k \cos(kx) + b_1 \sin(x) + \cdots + b_j \cos(jx))\).
5. Prove that there is no function \(f \in L^1_{\text{loc}}(\mathbb{R})\) such that \(\delta_0(\phi) = \phi(0) = \int_{\mathbb{R}} f \phi\).
6. Let \(p > 1, q > 1\) be such that \((1/p) + (1/q) = 1\). Prove that if \(f \in L^p(X), g \in L^q(X)\) then

\[
| \int_X f \cdot g | \leq \left( \int_X |f|^p \right)^{1/p} \left( \int_X |g|^q \right)^{1/q}.
\]

Due Tuesday, February 10, 2009 by 5:00pm.

Homework No. 3
1. Prove that the set of functions of the form
\[
f(x,y) = \sum_{i,j=0,\ldots,k} (a_{ij} \sin(ix) \sin(jy) + b_{ij} \sin(ix) \cos(jy) + c_{ij} \cos(ix) \sin(jy) + d_{ij} \cos(ix) \cos(jy)), \quad a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}
\]
is dense in \(L^2((0, 2\pi) \times (0, 2\pi))\).
2. Let \(s : \mathbb{R} \to \mathbb{R}\) be locally integrable and \(f(x, y) = s(x + y)\). Let \(T\) be the distribution given by \(f\). Prove that \(T_{yy} - T_{xx} = 0\).
3. Let \( u(x, y) = 1 \) if \( x > 0 \) and \( y > 0 \), and \( u(x, y) = 0 \) otherwise. Prove that, as a distribution,

\[
\frac{\partial^2 u}{\partial x \partial y} = \delta_{(0,0)}.
\]

4. Let \( \phi \in \hat{C}^\infty(a, b) \) prove that \( \phi = \psi' \) for some \( \psi \in \hat{C}^\infty(a, b) \) if and only if \( \int_a^b \phi(x)dx = 0 \).

5. Let \( T \) be a distribution in \((a, b)\). Prove that if \( T' = 0 \) then there is a real number \( k \) such that \( T(\phi) = \int_a^b k \phi \), i.e., \( T \) is a constant function. Hint: Let \( w \in \hat{C}^\infty(a, b) \) be such that \( \int_a^b w = 1 \), and consider \( \psi = \phi - w \int_a^b \phi \).

Due February 18, 2009 by 2:00pm.

Homework No. 5

1. Let \( \Omega = (0, \pi) \times (0, \pi) \) and \( \{f_n\}_n \) a sequence of functions in \( L^2(\Omega) \) that converges to 0. Prove that if \( \Delta u_n = f_n \) in \( \Omega \), \( u_n = 0 \) on \( \partial \Omega \) then \( \{u_n\}_n \) converges to zero in \( H^{2,2}(\Omega) \).

2. Let \( \Omega = (0, \pi) \) and \( \{h_n\}_n \) a sequence of functions in \( L^2(\Omega) \) that converges to 0. Prove that if \((u_n)_t = (u_n)_{xx}, u_n(x, 0) = h_n(x), \) and \( u_n(0, t) = u_n(\pi, t) \), then \( \{u_n(\cdot, t)\}_n \) converges to zero in \( H^k(\Omega) \) for all \( k > 0 \) and all \( t > 0 \).

3. Let \( \Omega = (0, \pi) \) and \( h \in L^2(\Omega) \). Prove that for each \( f \in L^2(\Omega \times [0, T]) \) the heat equation \( u_t = u_{xx} + f, u(x, 0) = h(x) \), and \( u(0, t) = u(\pi, t) = 0 \) has a unique solution. Analyze the regularity of such a solution in terms of \( f \).

4. Let \( \Omega = (0, \pi) \times (0, \pi), f \in C^1(\Omega) \) and \( v(x) = v(x, 0) \). Prove that

\[
\int_0^\pi v^2(x)dx \leq \|f\|^2_{H^{1 - 2}(\Omega)}
\]

What can you say about \( v_1(y) = f(0, y), v_2(x) = f(x, \pi) \), \( v_3(y) = f(1, y) \)? Hint: Express \( f \) is Fourier series.

Due March 4, 2009 by 2:00pm.
then
\[ \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \right) = f(x) \]
in the sense of distributions.

3. Let \( a_{ij} \) be as in problem 2. Suppose further that there exists \( c > 0 \) such that
\[ \sum_{i,j=1}^{N} a_{ij}(x)\alpha_i\alpha_j \geq c \sum_{i,j=1}^{N} \alpha_i^2 \text{ for all } (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^N, \ x \in \Omega. \]
Let
\[ < u, v >_3 = \int_{\Omega} \left( \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) \right) dx \]
Prove that \( < \cdot, \cdot >_3 \) defines an inner product in \( H = \mathbb{H}^{1,2}(\Omega) \). Prove that the norm defined by this inner product is equivalent to the norms in problem 1. That is a sequence converges in one norm if and only if it converges in the other two norms.

4. With the above notations prove that for all \( f \in L^2(\Omega) \) the equation (8) has a unique solution in \( H^1(\Omega) \).

Due March 11, 2009 by 2:00pm.

Homework No. 6
1. Let \( H \) be a real separable Hilbert space. Prove that if \( \{ u_k \}_k \subset H \) is a weakly convergent sequence then \( \{ u_k \}_k \) is bounded.
2. Let \( \{ u_1, u_2, \ldots \} \) be an orthonormal set. Let \( x_i = \sum_{j=1}^{\infty} a_{ij} u_j \). Prove that if \( \lim_{i \to \infty} a_{ij} \) exists for each \( j = 1, 2, \ldots \) and \( \{ x_i \}_i \) is bounded then \( \{ x_i \}_i \) converges weakly. Give an example where this convergence is not strong.
3. Let \( \Omega \) be bounded region in \( \mathbb{R}^N \), \( H = \mathbb{H}^{1,2}(\Omega) \), and \( J(u) = \int_{\Omega} u^2(x) dx \) for \( u \in H \). Let \( \phi_1 \) be such that \( J(\phi_1) = \max \{ J(u); \|u\|_H \leq 1 \} \) and \( M = \{ u \in H; < u, \phi_1 > = 0 \} \). Prove that there exists \( \phi_2 \in M \) such that \( J(\phi_2) = \max \{ J(u); u \in M, \|u\|_H \leq 1 \} \).
4. Let \( \phi_2 \) be as in problem 3 and \( \lambda_2 = (1/J(\phi_2)) \). Prove that \( -\Delta \phi_2 = \lambda_2 \phi_2 \), in the sense of distributions.
5. Inductively extend this process to build a sequence \( \{ \phi_1, \phi_2, \ldots \} \). What equation do the \( \phi_i \)'s satisfy?
Due March 25, 2009 by 2:00pm.

Homework No. 7
1. Let \( \lambda_i, \phi_i \) be as in homework 6. Let \( X \) be the subspace spanned by \( \{ \phi_1, \ldots, \phi_k \} \).
Prove that
\[ \int_{\Omega} \| \nabla u(x) \|^2 dx \leq \lambda_k \int_{\Omega} u^2(x) dx, \]
for all \( u \in X \).
2. Let \( Y = \{ u \in H; < y, x >_H = 0 \text{ for all } x \in X \} \). Prove that
\[ \int_{\Omega} \| \nabla y(x) \|^2 dx \geq \lambda_k \int_{\Omega} u^2(x) dx. \]
for all \( y \in Y \).
3. Prove that $\phi_i$ does not change sign in $\Omega$ if and only if $i = 1$. Feel free to use that if $f(t)$ is a Lipschitzian function (i.e $|f(s) - f(t)| \leq M|s - t|$ for all $s, t$ and some $M$) and $u \in H$ the $f \circ u \in H$.

4. Let $\Omega_1 \subset \Omega_2$ Prove that if $u \in \dot{H}^{1,2}(\Omega_1)$ then $u \in \dot{H}^{1,2}(\Omega_2)$.

5. Let $\lambda_1$ be the smallest eigenvalue of $-\Delta$ with zero boundary data in $\Omega_1$ and $\mu_1$ be the smallest eigenvalue of $-\Delta$ with zero boundary data in $\Omega_2$. Prove that $\mu_1 \leq \lambda_1$. What can you say about the first eigenvalue of $-\Delta$ with zero boundary data in a region contained in $(0, 1) \times (0, 1)$?

Due April 1, 2009 by 2:00pm.

Homework No. 8

1. Let $\Omega, \lambda_i, \phi_i$ be as in Homework No. 7. Study existence, uniqueness and regularity of the solutions to the parabolic problem

$$\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= \Delta u(x, t) + p(x, t) \quad x \in \Omega, t \in \mathbb{R}, \\
u(x, t) &= u(x, t + 2\pi) \quad x \in \Omega, t \in \mathbb{R}, u(x, t) = 0 \quad x \in \partial \Omega, t \in \mathbb{R}.
\end{align*}$$

(13)

Here $p(x, t)$ is $2\pi$-periodic in $t$ that belongs to $L^2(\Omega \times [0, 2\pi])$. Hint: Write $u$, and $p$ in the form $\sum_{i=1}^{\infty} a_i(t)\phi_i(x)$.

Due April 8, 2009 by 2:00pm.