A Bound on the Number of Spanning Trees in Bipartite Graphs

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Abstract

Richard Ehrenborg conjectured that in a bipartite graph G with parts X and Y, the number of spanning trees is at most $\prod_{v \in V(G)} \deg(v)$ divided by $|X| \cdot |Y|$. We make two main contributions. First, using techniques from spectral graph theory, we show that the conjecture holds for sufficiently dense graphs containing a cut vertex of degree 2. Second, using electrical network analysis, we show that the conjecture holds under the operation of removing an edge whose endpoints have sufficiently large degrees.

Our other results are combinatorial proofs that the conjecture holds for graphs having $|X| \leq 2$, for even cycles, and under the operation of connecting two graphs by a new edge.

We also make two new conjectures based on empirical data, each of which is stronger than Ehrenborg's conjecture.

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Chapter 1

Introduction

1.1 Motivation

We begin by motivating the notions of a *spanning tree* and a *bipartite graph*. A graph consists of vertices and edges, with each edge connecting two vertices. Graphs abstract real-world situations in which any two objects are either related or unrelated. For example, in the case of a social network, the objects are friends and the relation is mutual friendship.

Consider the following procedure: starting from a connected graph G, remove edges one at a time until no more edges can be removed without disconnecting the graph. The subgraphs of G that can be obtained from this procedure are known as *spanning trees* in G. In general, G will contain multiple spanning trees, obtained from G by removing different sets of edges. We denote by $\tau(G)$ the number of spanning trees in G.

To see an application of spanning trees, let G_1 be the graph in which vertices represent airports and every two vertices are connected by an edge. An airline wants passengers to be able to get from any airport to any other airport by taking a series of flights. To satisfy this constraint, the minimal sets of routes the airline can fly are exactly the spanning trees in G_1. This suggests that $\tau(G)$ tells us something about how connected G is. Indeed, $\tau(G)$ has been used as a measure of network reliability (Aggarwal and Rai, 1981).

We say a graph G is *bipartite* if its vertices can be divided into X and Y such that every edge of G connects a vertex in X to a vertex in Y. Bipartite graphs naturally model situations in which there are two classes of objects and only objects from different classes can be related. For example, to abstract ice cream flavor preferences of a group of people, we can define a bipartite graph G_2 in which some vertices represent people and the other vertices represent flavors. A person vertex and a flavor vertex are connected by an edge in G_2 exactly when that person enjoys ice cream of that flavor, and there are no other edges in G_2.

Our main goal in this thesis report is to explore a conjectured upper bound on the number of spanning trees $\tau(G)$ in a bipartite graph G. Though this bound is primarily of theoretical interest, it finds application in the design of experiments, where maximizing $\tau(G)$ in a bipartite graph is related to finding an optimum design (Cheng, 1981).

1.2 Ehrenborg's conjecture

Ehrenborg and van Willigenburg (2004) define a class of bipartite graphs called *Ferrers graphs*, and they prove that for a Ferrers graph G with parts X and Y,

$$\tau(G) = \frac{1}{|X| \cdot |Y|} \prod_{v \in V(G)} d_v, \tag{1.1}$$

where d_v denotes the degree of vertex v.

This report explores a conjecture due to Richard Ehrenborg (personal communication) that the right-hand side of Equation 1.1 gives an upper bound on $\tau(G)$ for every bipartite graph.

Conjecture 1.1. *Let G be a bipartite graph with parts X and Y. Then*

$$\tau(G) \leq \frac{1}{|X| \cdot |Y|} \prod_{v \in V(G)} d_v.$$

To our knowledge, the only previous work directly addressing Conjecture 1.1 is the preprint of Garrett and Klee (2014). We review their main results here. They show that Conjecture 1.1 is equivalent to the nonnegativity of a certain multivariate polynomial whose complexity depends only on the size of $|X|$. Using this polynomial, they computationally verified the conjecture for $|X| \leq 5$, which implies the following result.

Theorem 1.2. *Conjecture* 1.1 *holds for bipartite graphs with at most* 11 *vertices.*

Moreover, Garrett and Klee (2014) show that Conjecture 1.1 holds under the operation of adding a new vertex and connecting the new vertex to an existing vertex by a new edge.

Proposition 1.3. *Let G be a connected bipartite graph for which Conjecture* 1.1 *holds. Let $v \in V(G)$ and $u \notin V(G)$, and define the graph H by $V(H) = V(G) \cup \{u\}$ and $E(H) = E(G) \cup \{uv\}$. Then the conjecture holds for H also.*

Because any tree can be built up from a single vertex by iteratively adding leaf vertices, we have the following corollary.

Corollary 1.4. *Conjecture* 1.1 *holds for trees.*

1.3 Summary of contributions

Spectral graph theory is a branch of mathematics that associates matrices to a graph and studies how the spectra of these matrices relate to the graph's properties. For example, from the spectrum of a graph G's Laplacian matrix, Mohar (1997) derives bounds for the weight of an edge cut in G and the rate of convergence of a random walk on G.

While our main goal in this report is to explore Conjecture 1.1, we intend Chapter 3 as an expository introduction to spectral graph theory. The chapter requires only elementary graph theory and linear algebra. By providing details often missing from more advanced treatments such as those in Brouwer and Haemers (2012) and Chung (1997), the chapter serves as a brief undergraduate-level introduction to spectral graph theory.

The rest of this report is structured as follows. Chapter 2 defines notation and reviews some facts from graph theory and linear algebra. Chapter 3 introduces the fundamentals of spectral graph theory. In Chapter 4, after reviewing the theory of electrical networks, we follow the proof of Ehrenborg and van Willigenburg (2004) that the conjecture holds with equality for Ferrers graphs.

We present original results about Conjecture 1.1 in Chapter 5. Specifically, we contribute proofs that the conjecture holds:

- for graphs having $|X| \leq 2$,
- for even cycles,
- under the operation of connecting two graphs by a new edge,
- for sufficiently dense graphs containing a cut vertex of degree 2, and
- under the operation of removing an edge whose endpoints have sufficiently large degrees.

In Chapter 6, we make two new conjectures based on empirical data, among other suggestions for future work. Proving either conjecture of ours would establish Conjecture 1.1 as a consequence.

Chapter 2

Preliminaries

In this chapter, we define our notation and review some facts from elementary graph theory and linear algebra. These facts will be useful when spectral graph theory is introduced in the next chapter.

2.1 Graph theory

Let G be a graph. We denote its vertex set by $V(G)$ and its edge set by $E(G)$. A graph G is *finite* if $V(G)$ and $E(G)$ are finite sets. A graph G is *simple* if it is undirected, each edge connects two distinct vertices, and every two vertices are connected by at most one edge. Henceforth, when we say *graph*, we mean a finite simple graph.

From the above definition of a graph, we may obtain the definition of a *multigraph* by removing the requirement that every two vertices are connected by at most one edge. Thus, a multigraph can contain parallel edges (but, like graphs, cannot contain loops).

Let G be a graph on n vertices. We write $V(G) = \{1, 2, \ldots, n\}$ for convenience, so that each vertex is a positive integer. If vertices x and y are adjacent in G, we can denote that by $x \sim y$; if they are not adjacent in G, we can denote that by $x \nsim y$.

The *complete graph* on n vertices, denoted K_n, is the graph in which every two distinct vertices are adjacent.

A graph G is *bipartite* if $V(G)$ can be partitioned into sets X and Y such that every edge of G connects a vertex in X to a vertex in Y. If this is the case, then X and Y are called the *parts* of G.

Let G be a bipartite graph with parts X and Y. Then G is *complete bipartite* if $x \sim y$ for all $x \in X$ and $y \in Y$. If this is the case, we may refer to G as $K_{p,q}$ or $K_{q,p}$, where $p = |X|$ and $q = |Y|$.

A multigraph T is a *tree* if it is connected and contains no cycles.

Let G be a multigraph. We say a subgraph H of G *spans* G if H contains every vertex of G. A subgraph T of G is a *spanning tree* in G if T is a tree that spans G. We denote by $\tau(G)$ the number of spanning trees in G.

Let G be a graph. The *complement* of G, denoted G^c, is the graph satisfying: (i) $V(G^c) = V(G)$ and (ii) for all distinct $u, v \in V(G^c)$, we have $uv \in E(G^c)$ if and only if $uv \notin E(G)$.

Let G be a multigraph containing the edge $e = uv$. We can delete e from G; we denote the resulting multigraph by $G\backslash e$. We define a new multigraph, denoted G/e, which is obtained from G by *contracting* e. The vertex set $V(G/e)$ is obtained from $V(G)$ by replacing u and v with a new vertex w. The edge set $E(G/e)$ is obtained from $E(G)$ by deleting e, and then replacing each edge of the form ux or vx with a new edge wx. Note that G/e might not be a graph even if G is a graph.

We can count spanning trees using the following recurrence.

Proposition 2.1. *Let G be a multigraph containing the edge $e = uv$. Then*

$$\tau(G) = \tau(G\backslash e) + \tau(G/e).$$

Proof. Let T be a spanning tree in G. Exactly one of the following holds:

- T does not contain the edge e. Then deleting e doesn't change T or $V(G)$, so T is a spanning tree in $G\backslash e$. Since T already contains a uv-path that does not use $e = uv$, contracting e creates a cycle in T, so T is not a tree in G/e.

- T contains the edge e. Then T is a spanning tree in G/e, but deleting e disconnects T, so T is not a tree in $G\backslash e$.

This defines a function f that maps each spanning tree in G to a spanning tree in either $G\backslash e$ or G/e (but not both). It is easy to see that f is bijective, from which the result follows. □

Let G be a graph containing the vertex v. We denote by $N(v)$ the set of vertices in $V(G)$ adjacent to v.

Let G be a multigraph containing the vertex v. The *degree* of v, denoted d_v, is the number of edges in G incident to v. We may instead write $d_v(G)$ when we wish to specify the graph. The vertex v is *isolated* if $d_v = 0$.

Let G be a graph and $V_0 \subseteq V(G)$. The *volume* of V_0, denoted $\operatorname{vol} V_0$, is the sum $\sum_{v \in V_0} d_v$. The volume of $V(G)$ can also be denoted $\operatorname{vol} G$.

Let G and H be graphs with disjoint vertex sets. The *union* of G and H, denoted $G \cup H$, is defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

Let G be a graph. A *matching* in G is a subset of $E(G)$ in which no two edges are incident to the same vertex.

2.2 Linear algebra

Let M be a matrix. We denote by $M(i, j)$ the (i, j)-th entry of M. We denote by $M[i, j]$ the matrix obtained from M by deleting row i and column j. The transpose of M is denoted by $M^\top$, and the conjugate transpose of M is denoted by M^*.

We denote by $\operatorname{diag}(x_1, \dots, x_n)$ the $n \times n$ diagonal matrix M with diagonal entries $m_{ii} = x_i$. We denote by I_n the $n \times n$ identity matrix; we will write I for I_n when n is clear from context.

Now let M be an $n \times n$ matrix. We denote by M^S the matrix obtained from M by deleting the rows and columns indexed by $S \subseteq \{1, \dots, n\}$. For example, we have $M^{\{i\}} = M[i, i]$.

The *determinant* of M, denoted $\det M$, can be defined as

$$\det M = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} M\big(i, \sigma(i)\big),$$

where S_n denotes the symmetric group of order n and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ.

The *characteristic polynomial* of M, denoted p_M, is the polynomial defined by

$$p_M(\lambda) = \det(\lambda I - M).$$

We have $p_M(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i)$, where the λ_i's are the eigenvalues of M.

The (i, j)-*cofactor* of M, denoted $c_{ij}(M)$, is defined as

$$c_{ij}(M) = (-1)^{i+j} \det M[i, j].$$

The cofactors of M are involved in *cofactor expansion*, a procedure to compute the determinant of M. Cofactor expansion relies on the fact that

$$\det M = \sum_{j=1}^{n} M(p, j) \cdot c_{pj}(M) = \sum_{i=1}^{n} M(i, q) \cdot c_{iq}(M)$$

for all $1 \leq p, q \leq n$.

Using cofactor expansion, we show that the derivative of $p_M(\lambda)$ can be expressed in terms of the characteristic polynomials of certain submatrices of M.

Lemma 2.2. *Let M be an $n \times n$ real matrix. Then*

$$p'_M(\lambda) = \sum_i p_{M^{\{i\}}}(\lambda).$$

Proof. Let $r_i(\lambda)$, a function of λ, denote the ith row of $\lambda I - M$. It can be shown that the determinant of $\lambda I - M$ is a multilinear function f of its rows, that is,

$$p_M(\lambda) = \det(\lambda I - M) = f(r_1, r_2, \ldots, r_n).$$

Multiplying a row by a scalar c also multiplies the determinant by c. This fact, together with the multivariable chain rule, gives

$$p'_M(\lambda) = f(r'_1, r_2, \ldots, r_n) + f(r_1, r'_2, \ldots, r_n) + \cdots + f(r_1, r_2, \ldots, r'_n);$$

for more details see Hanche-Olsen (2012). Let M_i be the matrix whose rows are $r_1, \ldots, r'_i, \ldots, r_n$, so that $\det M_i = f(r_1, \ldots, r'_i, \ldots, r_n)$ and

$$p'_M(\lambda) = \sum_i \det M_i. \tag{2.1}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}(\lambda I - L) = I_n$, the row vector r'_i has 1 in the ith position and 0 in every other position. Using cofactor expansion along row i, we compute

$$\det M_i = \det\left(\lambda I - M^{\{i\}}\right) = p_{M^{\{i\}}}(\lambda).$$

Substituting this into Equation 2.1 gives the result. □

The matrices we will associate to a disconnected graph are block diagonal, with each block corresponding to a component of the graph. The spectrum of a block diagonal matrix is determined by the spectra of its blocks.

Proposition 2.3. *Let M be a block diagonal matrix with blocks $M_1, M_2, \ldots, M_k$, so that*

$$M = \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_k \end{bmatrix}.$$

Then the eigenvalues of M are the union (including multiplicities) of the eigenvalues from each block M_i.

2.2.1 Real symmetric matrices

The matrices we associate with a graph will always be real symmetric. We present the following properties of real symmetric matrices. In fact, these properties are enjoyed by the more general class of Hermitian matrices; proofs can be found in Horn and Johnson (2013: Theorems 2.5.6 and 2.5.3).

Theorem 2.4. *Let M be an $n \times n$ real symmetric matrix and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of M. Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then*:

(*i*) *$\lambda_1, \ldots, \lambda_n$ are real.*

(*ii*) *$M = U\Lambda U^*$ for some $n \times n$ unitary matrix U.*

(*iii*) *M has n orthonormal eigenvectors.*

Let M be a real symmetric matrix. Then M is diagonalizable by Theorem 2.4, so the algebraic multiplicity of each eigenvalue of M equals its geometric multiplicity. Thus, we can refer to the *multiplicity* of an eigenvalue of M without being ambiguous.

We can characterize each eigenvalue of a real symmetric matrix in terms of the following function.

Definition. *Let M be an $n \times n$ real symmetric matrix. The* Rayleigh quotient *of M, denoted R_M, is the function from $\mathbb{R}^n \backslash \{0\}$ to $\mathbb{R}$ defined by*

$$R_M(\mathbf{f}) = \frac{\mathbf{f}^\top M \mathbf{f}}{\mathbf{f}^\top \mathbf{f}}\,.$$

The following result is a version of the Courant–Fischer theorem (Horn and Johnson, 2013: Theorem 4.2.6).

Theorem 2.5. *Let M be an $n \times n$ real symmetric matrix, and let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of M. For each $1 \leq i \leq n$, let $\mathbf{f}_i$ be an eigenvector corresponding to λ_i. Define U_k as the span of $\{\mathbf{f}_i : 1 \leq i < k\}$, and define W_k as the span of $\{\mathbf{f}_i : k < i \leq n\}$. Then*

$$\lambda_k = \min_{\mathbf{f} \in U_k^\perp \backslash \{0\}} R_M(\mathbf{f}) \tag{2.2}$$

and

$$\lambda_k = \max_{\mathbf{f} \in W_k^\perp \backslash \{0\}} R_M(\mathbf{f}) \tag{2.3}$$

for each $1 \leq k \leq n$.

Proof. Choose an orthonormal set $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of eigenvectors of M such that $\mathbf{e}_i$ corresponds to λ_i and the span of $\{\mathbf{e}_i : 1 \leq i \leq k-1\}$ equals U_k. First, we show that Equation 2.2 holds by showing that $R_M(\mathbf{e}_k) = \lambda_k$ and $R_M(\mathbf{f}) \geq \lambda_k$ for all $\mathbf{f} \in U_k^{\perp} \backslash \{0\}$. We find

$$R_M(\mathbf{e}_k) = \frac{\mathbf{e}_k^{\top} \lambda_k \mathbf{e}_k}{\mathbf{e}_k^{\top} \mathbf{e}_k} = \lambda_k.$$

It remains to show that $R_M(\mathbf{f}) \geq \lambda_k$ for an arbitrary $\mathbf{f} \in U_k^{\perp} \backslash \{0\}$. Write $\mathbf{f} = \sum_{i=k}^{n} \alpha_i \mathbf{e}_i$. Using the fact that the $\mathbf{e}_i$'s are orthonormal, the numerator of $R_M(\mathbf{f})$ is

$$\begin{aligned}
\mathbf{f}^{\top} M \mathbf{f} &= \left(\sum_{i=k}^{n} \alpha_i \mathbf{e}_i \right)^T \left(\sum_{j=k}^{n} \alpha_j M \mathbf{e}_j \right) \\
&= \left(\sum_{i=k}^{n} \alpha_i \mathbf{e}_i \right)^T \left(\sum_{j=k}^{n} \alpha_j \lambda_j \mathbf{e}_j \right) \\
&= \sum_{i=k}^{n} \sum_{j=k}^{n} \alpha_i \alpha_j \lambda_j \mathbf{e}_i^{\top} \mathbf{e}_j \\
&= \sum_{i=k}^{n} \alpha_i^2 \lambda_i,
\end{aligned}$$

and the denominator of $R_M(\mathbf{f})$ is

$$\mathbf{f}^{\top} \mathbf{f} = \left(\sum_{i=k}^{n} \alpha_i \mathbf{e}_i \right)^{\top} \left(\sum_{j=k}^{n} \alpha_j \mathbf{e}_j \right) = \sum_{i=k}^{n} \sum_{j=k}^{n} \alpha_i \alpha_j \mathbf{e}_i^{\top} \mathbf{e}_j = \sum_{i=k}^{n} \alpha_i^2.$$

Thus

$$R_M(\mathbf{f}) = \frac{\sum_{i=k}^{n} \alpha_i^2 \lambda_i}{\sum_{i=k}^{n} \alpha_i^2}.$$

This is a weighted average of the eigenvalues $\lambda_k, \ldots, \lambda_n$, where the weight of λ_i is α_i^2. Because these weights are nonnegative, it is easy to show that

$$R_M(\mathbf{f}) \geq \min_{k \leq i \leq n} \lambda_i = \lambda_k,$$

so Equation 2.2 holds.

To prove Equation 2.3, we apply Equation 2.2 to the matrix $-M$. Write the eigenvalues of $-M$ as $\lambda_1(-M) \leq \cdots \leq \lambda_n(-M)$. Because $-M$ has eigenvalues $-\lambda_1, \ldots, -\lambda_n$, we have $\lambda_{n+1-k}(-M) = -\lambda_k$, and furthermore $\mathbf{f}_k$ is an eigenvector corresponding to $\lambda_{n+1-k}(-M)$. Now by Equation 2.2,

$$\lambda_{n+1-k}(-M) = \min_{\mathbf{f} \in W_k^{\perp} \setminus \{0\}} R_{-M}(\mathbf{f}).$$

Equation 2.3 now follows from the computation

$$\lambda_k = -\lambda_{n+1-k}(-M) = \max_{\mathbf{f} \in W_k^{\perp} \setminus \{0\}} -R_{-M}(\mathbf{f}) = \max_{\mathbf{f} \in W_k^{\perp} \setminus \{0\}} R_M(\mathbf{f}),$$

where the last equality holds because $R_{-M}(\mathbf{f}) = -R_M(\mathbf{f})$. □

Note that $U_1^{\perp} = \mathbb{R}^n = W_n^{\perp}$ in the statement of the preceding theorem, since U_1 and W_n are each the zero subspace. Thus, we have the following characterization of the smallest and largest eigenvalues.

Corollary 2.6. *Let M be an $n \times n$ real symmetric matrix, and let $\lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of M. Then*

$$\lambda_1 = \min_{\mathbf{f} \in \mathbb{R}^n \setminus \{0\}} R_M(\mathbf{f})$$

and

$$\lambda_n = \max_{\mathbf{f} \in \mathbb{R}^n \setminus \{0\}} R_M(\mathbf{f}).$$

For example, consider the real symmetric matrix

$$M = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

with eigenvalues $\lambda_1 \leq \lambda_2$. Writing $\mathbf{f} = (x, y)$, the Rayleigh quotient of M is the function from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}$ defined by

$$R_M(x, y) = \frac{x(x + 3y) + y(3x + y)}{x^2 + y^2}.$$

As (x, y) ranges over $\mathbb{R}^2 \setminus \{0\}$, the quantity $R_M(x, y)$ is minimized at $\lambda_1 = -2$ when $x = -y$ and maximized at $\lambda_2 = 4$ when $x = y$, which verifies Corollary 2.6 for this particular matrix M.

2.2.2 Positive semidefinite matrices

If a real symmetric matrix M satisfies $\mathbf{f}^\top M\mathbf{f} \geq 0$ for all $\mathbf{f} \in \mathbb{R}^n$, we say it is *positive semidefinite*. We show that every eigenvalue of such a matrix is nonnegative.

Proposition 2.7. *Let M be a positive semidefinite matrix. Then every eigenvalue of M is nonnegative.*

Proof. Let λ be an eigenvalue of M. Since M is real symmetric, we know λ is real by Theorem 2.4. Let $\mathbf{f} \in \mathbb{R}^n$ be an eigenvector of M corresponding to λ. We compute

$$\lambda\|\mathbf{f}\|^2 = \lambda\mathbf{f}^\top\mathbf{f} = \mathbf{f}^\top\lambda\mathbf{f} = \mathbf{f}^\top M\mathbf{f} \geq 0.$$

Since $\mathbf{f}$ is nonzero, we have $\|\mathbf{f}\|^2 > 0$ and so $\lambda \geq 0$. □

Chapter 3

Spectral graph theory

In this chapter, we introduce spectral graph theory by defining two matrices commonly associated with a graph: the Laplacian and the normalized Laplacian. We build up to a proof of Kirchhoff's theorem (Theorem 3.18), which counts spanning trees in terms of the Laplacian eigenvalues.

3.1 The Laplacian

Given a graph G on n vertices, we can represent all of the graph's structure in an $n \times n$ matrix whose rows and columns are indexed by the vertex set $V(G) = \{1, \ldots, n\}$.

Definition. *Let G be a graph on n vertices. The* adjacency matrix *of G, denoted $A(G)$, is the $n \times n$ matrix whose (i, j)-th entry is* 1 *if vertices i and j are adjacent, and* 0 *otherwise. We will write A for $A(G)$ when G is clear from context.*

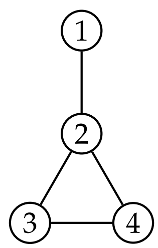

Figure 3.1 The example graph H_1.

For example, consider the graph H_1 in Figure 3.1. Its adjacency matrix is the 4×4 matrix

$$A(G_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

We can also record the vertex degrees of G in an $n \times n$ matrix. Subtracting the adjacency matrix from this gives a matrix which is immensely useful in spectral graph theory.

Definition. *Let G be a graph on n vertices. The* degree matrix *of G, denoted $D(G)$, is the $n \times n$ diagonal matrix whose (i, i)-th entry is the degree of vertex i. The* Laplacian *of G, denoted $L(G)$, is defined as $L(G) = D(G) - A(G)$. We will write D for $D(G)$ and L for $L(G)$ when G is clear from context.*

For example, the graph H_1 in Figure 3.1 has degree matrix

$$D(G_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and Laplacian

$$L(G_1) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}.$$

The Laplacian can also be described entrywise as

$$L(i, j) = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Since D and A are always real symmetric, the Laplacian is also real symmetric. By Theorem 2.4, the Laplacian eigenvalues are real, so they can be ordered. We elect to index these eigenvalues from zero, so that μ_0 denotes the smallest eigenvalue.

Definition. *Let G be a graph on n vertices. Let the eigenvalues of L be $\mu_0(G) \leq \mu_1(G) \leq \cdots \leq \mu_{n-1}(G)$. We will write μ_i for $\mu_i(G)$ when G is clear from context.*

For more information on the Laplacian and its eigenvalues, we refer the reader to the extensive surveys by Merris (1994) and Zhang (2011).

Let $\mathbf{f} \in \mathbb{R}^n$ be a vector. Recall that for a graph on n vertices, we write $V(G) = \{1, 2, \ldots, n\}$. This lets us consider the vector $\mathbf{f}$ as a function that assigns a real number to each vertex of G, and also lets us express the Laplacian as a quadratic form.

Proposition 3.1. *Let G be a graph on n vertices, and let $\mathbf{f} \in \mathbb{R}^n$. Then*

$$\mathbf{f}^\top L\mathbf{f} = \sum_{i\sim j} (f_i - f_j)^2,$$

where the sum is over all unordered pairs i, j of adjacent vertices.

Proof. Write $D = \operatorname{diag}(d_1, \ldots, d_n)$. We compute

$$\begin{aligned}
\mathbf{f}^\top L\mathbf{f} &= \mathbf{f}^\top D\mathbf{f} - \mathbf{f}^\top A\mathbf{f} \\
&= \sum_i f_i^2 d_i - \sum_i \sum_j f_i f_j a_{ij} \\
&= \frac{1}{2}\left(\sum_i f_i^2 d_i - 2\sum_i \sum_j f_i f_j a_{ij} + \sum_j f_j^2 d_j \right).
\end{aligned}$$

Since $d_i = \sum_j a_{ij}$ and $d_j = \sum_i a_{ij}$,

$$\mathbf{f}^\top L\mathbf{f} = \frac{1}{2}\sum_i \sum_j a_{ij}(f_i - f_j)^2 = \sum_{i\sim j}(f_i - f_j)^2. \qquad \square$$

We just showed that $\mathbf{f}^\top L\mathbf{f} \geq 0$ for all $\mathbf{f} \in \mathbb{R}^n$, so L is positive semidefinite. Thus, by Proposition 2.7, the eigenvalues of L are all nonnegative; equivalently, $\mu_0 \geq 0$. In fact, $\mu_0 = 0$ always.

Proposition 3.2. *Let G be a graph on n vertices. Then $\mu_0 = 0$, with the all-ones vector $\mathbb{1} \in \mathbb{R}^n$ as a corresponding eigenvector.*

Proof. In both D and A, the entries in row i sum to the degree of vertex i. Since $L = D - A$, the columns of L sum to the zero vector, so 0 is an eigenvalue of L with corresponding eigenvector $\mathbb{1}$. The result that $\mu_0 = 0$ then follows from the inequality $\mu_0 \geq 0$, which we have already shown. $\square$

A stronger result is true: the multiplicity of 0 as an eigenvalue of L counts the components of G. We first prove this for connected graphs.

Lemma 3.3. *Let G be a connected graph. Then* 0 *is an eigenvalue of L with multiplicity* 1.

Proof. From Proposition 3.2, we know 0 is an eigenvalue of L, so it remains to show that this eigenvalue 0 has multiplicity at most 1.

Suppose $\mathbf{f} \in \mathbb{R}^n$ is an eigenvector corresponding to the eigenvalue 0, so $\mathbf{f}^\top L\mathbf{f} = 0$. By Proposition 3.1, we have $\sum_{i\sim j}(f_i - f_j)^2 = 0$, which can only hold if $f_i = f_j$ for every pair i, j of adjacent vertices. Since G is connected, the eigenvector $\mathbf{f}$ must be a nonzero multiple of $\mathbb{1}$. Because $\mathbf{f}$ was an arbitrary eigenvector corresponding to 0, the eigenvalue 0 has multiplicity at most 1. □

Theorem 3.4. *Let G be a graph. Then the number of components of G equals the multiplicity of* 0 *as an eigenvalue of L.*

Proof. Let k be the number of components of G. Without loss of generality, we may assume that vertices are ordered so that each component's vertices are contiguous in the ordering. Then L has the block diagonal form

$$L = \begin{bmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{bmatrix},$$

where each L_i is a Laplacian for the i-th component. Now by Lemma 3.3, each L_i has 0 as an eigenvalue with multiplicity 1, so by Proposition 2.3, we conclude that L has 0 as an eigenvalue with multiplicity k. □

The Laplacian eigenvalues of a graph are related to those of its complement.

Theorem 3.5. *Let G be a graph on n vertices. Then the eigenvalues of G^c are* $\mu_0 = 0$ *and* $\mu_i(G^c) = n - \mu_{n-i}(G)$ *for* $1 \le i \le n-1$.

Proof. Let J denote the $n \times n$ all-ones matrix. Observe that

$$\begin{aligned} D(G) + D(G^c) &= D(K_n) = (n-1)I, \\ A(G) + A(G^c) &= A(K_n) = J - I. \end{aligned}$$

Thus, the Laplacian of G^c is

$$\begin{aligned} L(G^c) &= D(G^c) - A(G^c) \\ &= (n-1)I - D(G) + A(G) - J + I \\ &= nI - J - L(G). \end{aligned}$$

By Theorem 2.4, the eigenspaces of $L(G)$ are orthogonal. We know the all-ones vector $\mathbb{1} \in \mathbb{R}^n$ is an eigenvector of $L(G)$ by Proposition 3.2. Thus, we may find eigenvectors $\mathbf{f}_1, \ldots, \mathbf{f}_{n-1} \in \mathbb{R}^n$ of $L(G)$ such that $\{\mathbb{1}, \mathbf{f}_1, \ldots, \mathbf{f}_{n-1}\}$ is an orthogonal set.

Let $\mathbf{f} \in \{\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}\}$, and say $\mathbf{f}$ as an eigenvector of $L(G)$ corresponds to the eigenvalue μ. We know $\mathbf{f}$ is orthogonal to $\mathbb{1}$, so $J\mathbf{f} = 0$ and

$$\begin{aligned} L(G^c)\,\mathbf{f} &= nI\mathbf{f} - J\mathbf{f} - L(G)\,\mathbf{f} \\ &= n\mathbf{f} - 0 - \mu\mathbf{f} \\ &= (n - \mu)\mathbf{f}. \end{aligned}$$

Thus, $\mathbf{f}$ is an eigenvector of $L(G^c)$ that corresponds to the eigenvalue $n - \mu$. The result follows from noting that the set $\{\mathbb{1}, \mathbf{f}_1, \ldots, \mathbf{f}_{n-1}\}$ consists of eigenvectors of $L(G^c)$. □

The preceding result gives an upper bound on the largest Laplacian eigenvalue μ_{n-1}.

Corollary 3.6. *Let G be a graph. Then $\mu_{n-1} \leq n$, with equality if and only if G^c is disconnected.*

Proof. By Theorem 3.4, $\mu_1(G^c) \geq 0$, with equality if and only if G^c is disconnected. Using Theorem 3.5, we have

$$\mu_{n-1}(G) = n - \mu_1(G^c) \leq n. \qquad \square$$

3.2 The normalized Laplacian

We showed in the preceding section that $[0, n] \subseteq \mathbb{R}$ is the smallest interval that contains every Laplacian eigenvalue of every graph on n vertices. If two graphs have different numbers of vertices, then we may put these graphs' spectra on the same scale by working with a normalized version of the Laplacian.

Definition. *Let G be a graph on n vertices that has no isolated vertices. The* normalized Laplacian *of G, denoted $\mathcal{L}(G)$, is defined as*

$$\mathcal{L}(G) = D(G)^{-1/2} \cdot L(G) \cdot D(G)^{-1/2}.$$

We will write $\mathcal{L}$ for $\mathcal{L}(G)$ when G is clear from context.

Left-multiplication by $D^{-1/2}$ scales the rows of L; row i is scaled by $d_i^{-1/2}$. Similarly, right-multiplication by $D^{-1/2}$ scales the columns of L; column j is scaled by $d_j^{-1/2}$. Thus, $\mathcal{L}$ is related to L by

$$\mathcal{L}(i,j) = \frac{L(i,j)}{\sqrt{d_i d_j}}.$$

Equivalently, $\mathcal{L}$ can be described entrywise as

$$\mathcal{L}(i,j) = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{1}{\sqrt{d_i d_j}} & \text{if } i \neq j \text{ and } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

In this section, we present properties of the normalized Laplacian spectrum, mostly following the presentation in Chung (1997). For the remainder of this section, we assume that graphs have no isolated vertices, so that the matrices $D^{-1/2}$ and $\mathcal{L}$ are defined.

Like the Laplacian, the normalized Laplacian has real eigenvalues, so these eigenvalues can be ordered.

Definition. *Let G be a graph on n vertices. Let the eigenvalues of $\mathcal{L}$ be $\lambda_0(G) \leq \lambda_1(G) \leq \cdots \leq \lambda_{n-1}(G)$. We will write λ_i for $\lambda_i(G)$ when G is clear from context.*

These eigenvalues sum to the number of vertices in the graph.

Proposition 3.7. *Let G be a graph on n vertices. Then $\sum_{i=0}^{n-1} \lambda_i = n$.*

Proof. We have

$$\sum_{i=0}^{n-1} \lambda_i = \operatorname{tr} \mathcal{L} = n,$$

since the diagonal entries of $\mathcal{L}$ are 1. □

We can express the normalized Laplacian as a quadratic form, just like we did for the Laplacian (Proposition 3.1).

Proposition 3.8. *Let G be a graph on n vertices. Let $\mathbf{g} \in \mathbb{R}^n$ and define $\mathbf{f} = D^{-1/2}\mathbf{g}$. Then*

$$\mathbf{g}^\top \mathcal{L} \mathbf{g} = \sum_{i \sim j} (f_i - f_j)^2,$$

where the sum is over all unordered pairs i, j of adjacent vertices.

Proof. We compute

$$\mathbf{g}^\top \mathcal{L}\mathbf{g} = \mathbf{g}^\top D^{-1/2} L D^{-1/2}\mathbf{g} = \mathbf{f}^\top L\mathbf{f} = \sum_{i\sim j}(f_i - f_j)^2,$$

where the last equality is due to Proposition 3.1. □

We just showed that $\mathbf{g}^\top \mathcal{L}\mathbf{g} \geq 0$ for all $\mathbf{g} \in \mathbb{R}^n$, so $\mathcal{L}$ is positive semidefinite. Thus, by Proposition 2.7, we have $\lambda_0 \geq 0$. In fact, $\lambda_0 = 0$ always.

Proposition 3.9. *Let G be a graph on n vertices. Then $\lambda_0 = 0$, with $D^{1/2}\mathbb{1} \in \mathbb{R}^n$ as a corresponding eigenvector.*

Proof. We know $L\mathbb{1} = 0$ by Proposition 3.2, so $\mathcal{L}D^{1/2}\mathbb{1} = D^{-1/2}L\mathbb{1} = 0$. □

The following analog of Theorem 3.4 holds for $\mathcal{L}$.

Theorem 3.10. *Let G be a graph. Then the number of components of G equals the multiplicity of* 0 *as an eigenvalue of $\mathcal{L}$.*

Proof. Since $D^{-1/2}$ is a diagonal matrix without zeros on its diagonal, it has full rank. It follows that L and $\mathcal{L} = D^{-1/2}LD^{-1/2}$ have the same rank. By the rank-nullity theorem, L and $\mathcal{L}$ have the same nullity, and thus have 0 as an eigenvalue with the same multiplicity. The result then follows from Theorem 3.4. □

We can characterize the Rayleigh quotient $R_\mathcal{L}$ in terms of the graph's structure.

Proposition 3.11. *Let G be a graph on n vertices. Let $\mathbf{g} \in \mathbb{R}^n$ and define $\mathbf{f} = D^{-1/2}\mathbf{g}$. Then*

$$R_\mathcal{L}(\mathbf{g}) = \frac{\sum_{i\sim j}(f_i - f_j)^2}{\sum_i f_i^2 d_i}.$$

Proof. We compute

$$\mathbf{g}^\top \mathbf{g} = (D^{1/2}\mathbf{f})^\top (D^{1/2}\mathbf{f}) = \mathbf{f}^\top D\mathbf{f} = \sum_i f_i^2 d_i.$$

The result then follows by substituting this and the statement of Proposition 3.8 into the definition of $R_\mathcal{L}(\mathbf{g})$. □

Recall that our goal for a normalized version of the Laplacian was that graphs on different numbers of vertices would have spectra on the same scale. The next result shows that $\mathcal{L}$ as we defined it achieves this goal, because $[0, 2] \subseteq \mathbb{R}$ is the smallest interval that contains every normalized Laplacian eigenvalue of every graph on n vertices.

Theorem 3.12. *Let G be a connected graph on $n \geq 2$ vertices. Then $\lambda_{n-1} \leq 2$, with equality if and only if G is bipartite.*

Proof. For all $x, y \in \mathbb{R}$, we have the inequality

$$(x-y)^2 = 2\big(x^2+y^2\big) - (x+y)^2 \leq 2\big(x^2+y^2\big). \tag{3.1}$$

By Corollary 2.6 and then Proposition 3.11, the largest eigenvalue of $\mathcal{L}$ is

$$\lambda_{n-1} = \max_{\mathbf{g} \in \mathbb{R}^n \setminus \{0\}} R_{\mathcal{L}}(\mathbf{g}) = \max_{\mathbf{f} \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{i \sim j}(f_i - f_j)^2}{\sum_i f_i^2 d_i}.$$

Applying Inequality 3.1,

$$\frac{\sum_{i \sim j}(f_i - f_j)^2}{\sum_i f_i^2 d_i} \leq \frac{\sum_{i \sim j} 2\big(f_i^2 + f_j^2\big)}{\sum_i f_i^2 d_i} = 2.$$

We conclude that $\lambda_{n-1} \leq 2$ and that $\lambda_{n-1} = 2$ if and only if there exists $\mathbf{f} \in \mathbb{R}^n$ such that for every pair i, j of adjacent vertices,

$$(f_i - f_j)^2 = 2\big(f_i^2 + f_j^2\big),$$

which is equivalent to $f_i + f_j = 0$. It remains to show that this condition is satisfied if and only if G is bipartite.

($\Rightarrow$) Suppose there exists $\mathbf{f} \in \mathbb{R}^n$ such that $f_i + f_j = 0$ for every pair i, j of adjacent vertices. Since G is connected, there must be a partition of $V(G)$ into parts X, Y such that, for some constant $c \in \mathbb{R}$, we have

$$f_v = \begin{cases} c & \text{if } v \in X, \\ -c & \text{if } v \in Y \end{cases} \tag{3.2}$$

for every vertex v. Thus, we know no two vertices in X are adjacent because $c + c \neq 0$, and similarly no two vertices in Y are adjacent. It follows that G is bipartite with parts X and Y.

($\Leftarrow$) Suppose G is bipartite with parts X and Y. Define $\mathbf{f} \in \mathbb{R}^n$ by Equation 3.2. Then for every pair i, j of adjacent vertices, we have $i \in X$ and $j \in Y$ or vice versa, so $f_i + f_j = c - c = 0$. $\square$

Having characterized the smallest eigenvalue λ_0 and the largest eigenvalue λ_{n-1} of $\mathcal{L}$, we turn to the middle $n-2$ eigenvalues $\lambda_1, \ldots, \lambda_{n-2}$.

We will use the following result (Das et al., 2015: Theorem 3.4) without including a proof.

Theorem 3.13. *Let G be a graph on $n \geq 3$ vertices. Then the $n-2$ eigenvalues $\lambda_1, \ldots, \lambda_{n-2}$ are all equal if and only if G is complete or complete bipartite.*

For a bipartite graph G, Theorem 3.10 and Theorem 3.12 imply that $\lambda_0 + \lambda_{n-1} = 2$. In fact, if G is bipartite, then the entire spectrum of $\mathcal{L}$ is symmetric about 1.

Proposition 3.14. *Let G be a graph on $n \geq 2$ vertices. Then $\lambda_i + \lambda_{n-1-i} = 2$ for $0 \leq i \leq n-1$ if and only if G is bipartite.*

Proof. If G is not bipartite, then $\lambda_0 = 0$ by Theorem 3.10 and $\lambda_{n-1} < 2$ by Theorem 3.12, so $\lambda_0 + \lambda_{n-1} < 2$. Thus, we need only show that if G is bipartite, then $\lambda_i + \lambda_{n-1-i} = 2$ for $0 \leq i \leq n-1$.

Suppose G is bipartite with parts X and Y. Let $\mathbf{f} \in \mathbb{R}^n$ be an eigenvector of $\mathcal{L}$ corresponding to eigenvalue λ. Writing $\mathcal{L}\mathbf{f} = \lambda\mathbf{f}$ componentwise, for every vertex v we have

$$f_v + \sum_{u \sim v} \frac{-f_u}{\sqrt{d_u d_v}} = \lambda f_v,$$

or equivalently

$$(1-\lambda) f_v = \sum_{u \sim v} \frac{f_u}{\sqrt{d_u d_v}}.$$

Since G is bipartite, for every $x \in X$ and $y \in Y$ we have

$$(1-\lambda) f_x = \sum_{\substack{u \in Y \\ u \sim x}} \frac{f_u}{\sqrt{d_u d_x}}, \tag{3.3}$$

$$(1-\lambda) f_y = \sum_{\substack{u \in X \\ u \sim y}} \frac{f_u}{\sqrt{d_u d_y}}. \tag{3.4}$$

Define $\mathbf{g} \in \mathbb{R}^n$ by

$$g_v = \begin{cases} f_v & \text{if } v \in X, \\ -f_v & \text{if } v \in Y. \end{cases}$$

Substituting into Equations 3.3 and 3.4,

$$(1-\lambda)g_x = \sum_{\substack{u\in Y\\ u\sim x}} \frac{-g_u}{\sqrt{d_u d_x}},$$
$$(1-\lambda)(-g_y) = \sum_{\substack{u\in X\\ u\sim y}} \frac{g_u}{\sqrt{d_u d_y}}.$$

Equivalently,

$$g_v + \sum_{u\sim v} \frac{-g_u}{\sqrt{d_u d_v}} = (2-\lambda)g_v$$

for every vertex v, so $\mathbf{g}$ is an eigenvector of $\mathcal{L}$ corresponding to eigenvalue $2-\lambda$. It can be shown that the eigenvalues λ and $2-\lambda$ must have equal multiplicity, so the result follows. □

We have the following characterization of the smallest nontrivial eigenvalue of $\mathcal{L}$.

Proposition 3.15. *Let G be a graph on n vertices. Then*

$$\lambda_1 = \min_{\substack{\mathbf{f}\in\mathbb{R}^n\setminus\{0\}\\ \mathbf{f}\perp D\mathbb{1}}} \frac{\sum_{i\sim j}(f_i - f_j)^2}{\sum_i f_i^2 d_i}.$$

Proof. By Proposition 3.9, the vector $D^{1/2}\mathbb{1}$ is an eigenvector of $\mathcal{L}$ corresponding to eigenvalue $\lambda_0 = 0$. By Theorem 2.5,

$$\lambda_1 = \min_{\substack{\mathbf{g}\in\mathbb{R}^n\setminus\{0\}\\ \mathbf{g}\perp D^{1/2}\mathbb{1}}} R_{\mathcal{L}}(\mathbf{g}).$$

Note that $\mathbf{g} \perp D^{1/2}\mathbb{1}$ if and only if $D^{-1/2}\mathbf{g} \perp D\mathbb{1}$. The result then follows from Proposition 3.11. □

We can use the preceding result to put an upper bound on λ_1 by choosing an $\mathbf{f} \neq 0$ orthogonal to $D\mathbb{1}$. Chung (1997) gives the following variational characterization of λ_1, which removes the requirement that $\mathbf{f}$ be orthogonal to $D\mathbb{1}$. This is similar to a characterization of μ_1 by Mohar (1997: Proposition 2.7).

Proposition 3.16. *Let G be a graph on n vertices. Then*

$$\lambda_1 = \operatorname{vol} G \cdot \min \frac{\sum_{i\sim j}(f_i - f_j)^2}{\sum_{i,j}(f_i - f_j)^2 d_i d_j},$$

where the minimum is taken over all $\mathbf{f} \in \mathbb{R}^n$ such that $\mathbf{f}$ is not constant (that is, $\mathbf{f} \neq c\mathbb{1}$ for all $c \in \mathbb{R}$), and $\sum_{i,j}$ denotes the sum over all unordered pairs $\{i, j\}$ of vertices.

Proof. Suppose $\mathbf{f} \in \mathbb{R}^n$ satisfies $\mathbf{f} \perp D\mathbb{1}$. Then

$$\begin{aligned}
\sum_{i,j}(f_i - f_j)^2 d_i d_j &= \frac{1}{2}\sum_i \sum_j (f_i - f_j)^2 d_i d_j \\
&= \frac{1}{2}\sum_j d_j \cdot \sum_i f_i^2 d_i + \sum_i f_i d_i \cdot \sum_j f_j d_j + \frac{1}{2}\sum_i d_i \cdot \sum_j f_j^2 d_j \\
&= \left(\sum_i d_i\right)\left(\sum_i f_i^2 d_i\right) + \left(\sum_i f_i d_i\right)^2.
\end{aligned}$$

Rearranging while noting that $\sum_i f_i d_i = \mathbf{f}^\top D\mathbb{1} = 0$ and $\sum_i d_i = \operatorname{vol} G$,

$$\sum_i f_i^2 d_i = \frac{\sum_{i,j}(f_i - f_j)^2 d_i d_j}{\operatorname{vol} G}.$$

Substituting into Proposition 3.15,

$$\lambda_1 = \operatorname{vol} G \cdot \min_{\substack{\mathbf{f} \in \mathbb{R}^n \setminus \{0\} \\ \mathbf{f} \perp D\mathbb{1}}} \frac{\sum_{i\sim j}(f_i - f_j)^2}{\sum_{i,j}(f_i - f_j)^2 d_i d_j}.$$

Observe that the term being minimized is invariant under the operation of adding a constant vector to $\mathbf{f}$. Fix an arbitrary $\mathbf{g} \in \mathbb{R}^n$ that is not constant. The result would follow if there exists $\mathbf{h} \in \mathbb{R}^n$, obtained by adding a constant vector to $\mathbf{g}$, that satisfies $\mathbf{h} \neq 0$ and $\mathbf{h} \perp D\mathbb{1}$. We define such a vector $\mathbf{h}$ entrywise by

$$h_i = g_i - \frac{\sum_j g_j d_j}{\sum_j d_j}$$

for $1 \leq i \leq n$. If $\mathbf{h} = 0$, then every entry of $\mathbf{g}$ equals $\sum_j f_j d_j / \sum_j d_j$, contradicting the fact that $\mathbf{g}$ is not constant. Thus $\mathbf{h} \neq 0$. Finally, we compute

$$\begin{aligned}
\mathbf{h}^\top D\mathbb{1} &= \sum_i h_i d_i \\
&= \sum_i \left(g_i - \frac{\sum_j g_j d_j}{\sum_j d_j} \right) d_i \\
&= \sum_i g_i d_i - \sum_i d_i \cdot \frac{\sum_j g_j d_j}{\sum_j d_j} \\
&= 0,
\end{aligned}$$

showing that $\mathbf{h} \perp D\mathbb{1}$. □

3.3 Kirchhoff's theorem

Using a theorem of Kirchhoff (1847), we can count the spanning trees in any graph given only its Laplacian. In this section, we'll follow the proof of Kirchhoff's theorem given in Brouwer and Haemers (2012: Proposition 1.3.4), while adding considerable detail to their proof.

We generalize our definition of the Laplacian to include multigraphs.

Definition. *Let G be a multigraph on n vertices. Let $A(G)$ be the $n \times n$ matrix whose (i, j)-th entry is the number of edges between vertices i and j. Let $D(G)$ be the $n \times n$ diagonal matrix whose (i, i)-th entry is the degree of vertex i. The* Laplacian *of G, denoted $L(G)$, is defined as $L(G) = D(G) - A(G)$. We will write L for $L(G)$ when G is clear from context.*

This definition allows us to state the following necessary lemma for multigraphs. In the proof of this lemma, we will use the fact that Theorem 3.4 still holds for multigraphs.

Lemma 3.17. *Let G be a multigraph on n vertices. Then $\tau(G) = c_{xx}(L)$ for every vertex x of G.*

Proof. Fix a vertex x of G. We proceed by two-dimensional induction on n and on $d_x(G)$.

Suppose $n = 1$ and $d_x(G) = 0$. Then $\tau(G) = 1$ and, since the empty matrix has determinant 1, we have $c_{xx}(L) = 1$. This proves the result for $n = 1$ and $d_x(G) = 0$.

Suppose $n \geq 2$ and $d_x(G) = 0$. Then x is an isolated vertex, so $\tau(G) = 0$. The matrix $L^{\{x\}}$ is a Laplacian for the graph obtained from G by deleting x, so by Theorem 3.4, the matrix $L^{\{x\}}$ has 0 as an eigenvalue. Thus $c_{xx}(L) = \det L^{\{x\}} = 0$. This proves the result for $n \geq 2$ and $d_x(G) = 0$.

Having established the base cases where $d_x(G) = 0$, we proceed to the inductive step. Our inductive hypothesis will be that the result holds for any graph H that contains vertex x and satisfies

- H has fewer vertices than G, or
- H has as many vertices as G and $d_x(H) < d_x(G)$.

Suppose $d_x(G) \geq 1$. Let y be a vertex adjacent to x, and let e denote the edge xy. Consider $\det L^{\{x\}}$. Deleting e changes $L^{\{x\}}$ in only the (y, y)-th entry, which decreases by 1. Thus, computing $c_{xx}(L) = \det L^{\{x\}}$ by cofactor expansion along row y, we see that deleting e decreases $c_{xx}(L)$ by $\det L^{\{x,y\}}$, so

$$c_{xx}(L) = c_{xx}\big(L(G\backslash e)\big) + \det L^{\{x,y\}}.$$

Next, let us consider what happens to $\det L$ when the edge e is contracted. Contracting e is equivalent to:

1. replacing each edge zy with an edge zx, which changes only the rows and columns of L indexed by $\{x, y\}$, and then
2. deleting y, which deletes row y and column y of L.

Thus

$$c_{xx}\big(L(G/e)\big) = \det L^{\{x,y\}},$$

which gives

$$c_{xx}(L) = c_{xx}\big(L(G\backslash e)\big) + c_{xx}\big(L(G/e)\big).$$

Deleting e decreases $d_x(G)$ by 1 and does not change the number of vertices, so $c_{xx}(L(G\backslash e)) = \tau(G\backslash e)$ by the inductive hypothesis. Contracting e decreases the number of vertices by 1, so $c_{xx}(L(G/e)) = \tau(G/e)$ by the inductive hypothesis. Using Proposition 2.1,

$$\begin{aligned}
\tau(G) &= \tau(G\backslash e) + \tau(G/e) \\
&= c_{xx}\big(L(G\backslash e)\big) + c_{xx}\big(L(G/e)\big) \\
&= c_{xx}(L).
\end{aligned}$$

□

In the preceding proof of Lemma 3.17, even if G is a simple graph, the inductive hypothesis must hold for G/e, which need not be a simple graph. This forced us to establish the lemma for multigraphs.

We finally prove Kirchhoff's theorem, which extends Lemma 3.17 to the statement that *every* cofactor of L counts spanning trees.

Theorem 3.18 (Kirchhoff's theorem)**.** *Let G be a graph on n vertices. Then*

$$\tau(G) = c_{xy}(L) = \frac{1}{n}\prod_{i=1}^{n-1}\mu_i$$

for all vertices x, y of G.

Proof. Let x, y be vertices of G. Given Lemma 3.17, it suffices to prove that $c_{xx}(L) = c_{xy}(L)$ and that $\prod_{i=1}^{n-1}\mu_i = n \cdot \tau(G)$.

First, we prove that $c_{xx}(L) = c_{xy}(L)$. The columns of L sum to zero, so we can change $L(x, y)$ into $L(x, x)$ by performing the following elementary column operations:

1. adding each of the $n-2$ other columns to column x, then
2. multiplying column x by -1, and then
3. shifting column x to position y.

Operation 1 doesn't change the determinant, operation 2 multiplies the determinant by -1, and operation 3 multiplies the determinant by $(-1)^{x+y-1}$, so that

$$c_{xx}(L) = \det L(x, x) = (-1)^{x+y} \det L(x, y) = c_{xy}(L).$$

Second, we prove that $\prod_{i=1}^{n-1}\mu_i = n \cdot \tau(G)$. Since $\mu_0 = 0$, we have $p_L(\mu) = \mu\prod_{i=1}^{n-1}(\mu - \mu_i)$. Differentiating this product with respect to μ gives

$$p_L'(0) = (-1)^{n-1}\prod_{i=1}^{n-1}\mu_i.$$

On the other hand, using Lemma 2.2,

$$\begin{aligned} p_L'(0) &= \sum_x p_{L^{\{x\}}}(0) \\ &= \sum_x \det\big(-L^{\{x\}}\big) \\ &= (-1)^{n-1}\sum_x \det L^{\{x\}} \\ &= (-1)^{n-1}\sum_x c_{xx}(L), \end{aligned}$$

which gives $\prod_{i=1}^{n-1} \mu_i = \sum_x c_{xx}(L)$. Now $c_{xx}(L) = \tau(G)$ by Lemma 3.17 and the sum ranges over all n vertices, so

$$\prod_{i=1}^{n-1} \mu_i = \sum_x \tau(G) = n \cdot \tau(G). \qquad \square$$

Kirchhoff's theorem counts spanning trees using the eigenvalues of the Laplacian. We now prove a variation, due to Sachs (1976), that uses the eigenvalues of the normalized Laplacian instead.

Theorem 3.19. *Let G be a graph on n vertices that has no isolated vertices. Then*

$$\tau(G) = \frac{\prod_v d_v}{\sum_v d_v} \cdot \prod_{i=1}^{n-1} \lambda_i.$$

Proof. We will relate the $(n-1) \times (n-1)$ principal minors of L and $\mathcal{L}$. Let W be the set of all vertices except u, that is, $W = \{1, \dots, n\} \backslash \{u\}$. Let S_W be the symmetric group on W. Using the definitions of the determinant and the normalized Laplacian,

$$\begin{aligned} \det \mathcal{L}^{\{u\}} &= \sum_{\sigma \in S_W} \operatorname{sgn}(\sigma) \prod_{v \in W} \mathcal{L}\big(v, \sigma(v)\big) \\ &= \sum_{\sigma \in S_W} \operatorname{sgn}(\sigma) \prod_{v \in W} L\big(v, \sigma(v)\big) \cdot \big(d_v d_{\sigma(v)}\big)^{-1/2} \end{aligned} \tag{3.5}$$

For every permutation $\sigma \in S_W$,

$$\prod_{v \in W} d_v = \prod_{v \in W} d_{\sigma(v)},$$

since each vertex in W is counted once on each side. It follows that

$$\prod_{v \in W} \big(d_v d_{\sigma(v)}\big)^{-1/2} = \left(\prod_{v \in W} d_v\right)^{-1} = \frac{d_u}{\prod_v d_v},$$

which does not depend on σ. Substituting into Equation 3.5,

$$\det \mathcal{L}^{\{u\}} = \frac{d_u}{\prod_v d_v} \cdot \sum_{\sigma \in S_W} \operatorname{sgn}(\sigma) \prod_{v \in W} L\big(v, \sigma(v)\big) = \frac{d_u}{\prod_v d_v} \cdot \det L^{\{u\}},$$

so by Lemma 3.17,

$$\det \mathcal{L}^{\{u\}} = \frac{d_u}{\prod_v d_v} \cdot \tau(G). \tag{3.6}$$

Computations similar to those in the proof of Theorem 3.18 give

$$\prod_{i=1}^{n-1} \lambda_i = (-1)^{n-1} p'_{\mathcal{L}}(0) = \sum_u \det \mathcal{L}^{\{u\}}. \tag{3.7}$$

The result now follows by substituting Equation 3.6 into Equation 3.7 and then rearranging. $\square$

Chapter 4

Electrical networks

Ehrenborg and van Willigenburg (2004) show that Conjecture 1.1 holds with equality for *Ferrers graphs*, which are a class of bipartite graphs. In this chapter, we review the theory of electrical networks, follow their proof, and conclude by showing that Ferrers graphs are equivalent to connected *difference graphs*.

In this section, we view a connected graph G as a representation of an electrical network. Each edge ab of G is assumed to have a resistance $R(a,b)$. We may also denote the resistance of an edge e by $R(e)$.

Fix two distinct vertices of G: a source s and a sink t. We assume a unit current enters at s and leaves at t. Now for each edge ab of G, there is a current $I(a,b)$ in the edge and there is a voltage $V(a,b)$ across the edge. The voltage, resistance, and current are related by a physical law.

Ohm's law. *$V(a,b) = R(a,b) \cdot I(a,b)$ for each edge ab in G.*

Additionally, the currents and voltages must satisfy Kirchhoff's laws. Kirchhoff's current law states that the total current into any vertex equals the total current out of that vertex.

Kirchhoff's current law. *For every vertex v of G,*

$$\sum_{b \in N(v)} I(v,b) = 0.$$

Kirchhoff's voltage law states that the voltages around any cycle sum to zero.

Kirchhoff's voltage law. *For every cycle $C = v_1 \cdots v_k$ in G,*

$$\sum_{i=1}^{k} V(i, i+1) = 0,$$

where we set $v_{k+1} = v_1$.

4.1 Currents and spanning trees

The content in this section follows Bollobás (1979: Section II.1). Bollobás only proves Theorem 4.3 for the case where every edge has unit resistance, but we provide a proof for the general case. The *conductance* of an edge ab is $1/R(a, b)$, the reciprocal of its resistance. We will derive a formula for the current in each edge in terms of the conductances in spanning trees.

Definition. *The* weight *of a spanning tree T in G, denoted $w(T)$, is defined as*

$$w(T) = \left(\prod_{e \in E(T)} R(e) \right)^{-1}.$$

In the proof of the current formula, we will consider subgraphs of the electrical network G that are almost spanning trees.

Definition. *A* thicket *is a spanning subgraph of G with exactly two components U_s and U_t, such that U_s contains s and U_t contains t. The* weight *of a thicket U in G, denoted $w(U)$, is defined as*

$$w(U) = \left(\prod_{e \in E(U)} R(e) \right)^{-1}.$$

Definition. *Let ab be an edge of G. An ab-*thicket *is a thicket of G such that U_s contains a and U_t contains b. The* weight *of an ab-thicket U in G, denoted $w_{ab}(U)$, is defined as*

$$w_{ab}(U) = \left(R(a, b) \prod_{e \in E(U)} R(e) \right)^{-1}.$$

This next lemma immediately follows from the definitions of ab-thicket weight and thicket weight.

Lemma 4.1. *Let U be an ab-thicket. Then $w_{ab}(U) = w(U)/R(a,b)$.*

Note that any ab-thicket U is also a thicket, and by Lemma 4.1, U's weight as an ab-thicket differs from its weight as a thicket, unless the edge ab happens to have unit resistance.

Next, we relate the ab-thickets in the network G to some spanning trees in G. We will say an st-path P *contains* the edge ab if the sequence of vertices along P is $s \cdots ab \cdots t$.

Lemma 4.2. *Let ab be an edge of G. Let $W(a,b)$ be the sum of weights of all spanning trees in which the st-path contains ab. Let $W_U(a,b)$ be the sum of weights of all ab-thickets. Then $W(a,b) = W_U(a,b)$.*

Proof. Let S_T be the set of spanning trees considered in $W(a,b)$, and let S_U be the set of ab-thickets considered in $W_U(a,b)$. To show that the sum $W(a,b)$ of spanning tree weights equals the sum $W_U(a,b)$ of ab-thicket weights, it suffices to define a bijection $f : S_T \to S_U$ that preserves weight.

For a spanning tree $T \in S_T$, define $f(T)$ as the subgraph obtained from T by removing the edge ab. Since the st-path in T was $s \cdots ab \cdots t$, we know s, a are connected in $f(T)$ and b, t are connected in $f(T)$. Since there was only one st-path in T, the subgraph $f(T)$ has two components. It follows that $f(T)$ is an ab-thicket, and it's easy to see that f is bijective and that f preserves weight. □

We are now ready to derive the formula given in Kirchhoff (1847) for the current in each edge.

Theorem 4.3. *Let G be a connected graph representing an electrical network. Let W be the sum of weights of all spanning trees. Then the current in every edge ab of G is*

$$I(a,b) = \frac{W(a,b) - W(b,a)}{W},$$

where $W(a,b)$ is defined as in Lemma 4.2.

Proof. It is known that at most one set of currents satisfies both Kirchhoff's current law and Kirchhoff's voltage law, so we need only check that the currents defined satisfy both laws. Let T be an arbitrary spanning tree in G, and let P_T denote the (unique) st-path in T.

First, we check Kirchhoff's current law at vertices s and t. The path P_T contains the edge sb for exactly one $b \in N(s)$, so summing across all spanning trees T, we have $\sum_{b\in N(s)} W(s,b) = W$. The path P_T doesn't contain any edge bs where $b \in N(s)$, so $\sum_{b\in N(s)} W(b,s) = 0$. It follows that

$$\sum_{b\in N(s)} I(s,b) = \frac{W-0}{W} = 1,$$

so the current law holds at s. By a symmetric argument, the current law also holds at t.

Second, we check Kirchhoff's current law at a vertex $y \notin \{s,t\}$. If y is not on P_T, then T contributes 0 to the current into or out of y. Suppose y is on P_T, and say P_T is $s\cdots xyz\cdots t$. Then T contributes $w(T)$ to $W(x,y)$, and so contributes $w(T)$ to the current into y. Also, T contributes $w(T)$ to $W(y,z)$, and so contributes $w(T)$ to the current out of y. We have shown that an arbitrary spanning tree T contributes 0 to the net current flow out of y. It follows that the net current flow out of y is $\sum_{b\in N(y)} I(y,b) = 0$, so the current law holds at y.

Third, we check Kirchhoff's voltage law. Let C be an arbitrary cycle in G, and orient C's edges so that C is a directed cycle. It remains to check that the voltages around C sum to 0. Let U be an arbitrary thicket in G, with components U_s (containing s) and U_t (containing t). Using Lemma 4.2, we consider $W(a,b)$ in the formula for $I(a,b)$ as a sum of ab-thicket weights. Let ab be an arc in C. We check three cases:

Case 1: a,b are in the same component of U. Then U contributes 0 to $W(a,b)$ and $W(b,a)$, so U contributes 0 to the sum of voltages around C.

Case 2: a is in U_s and b is in U_t. Then the thicket U is also an ab-thicket, so U contributes $w_{ab}(U)/W$ to $I(a,b)$. By Ohm's law and Lemma 4.1, U contributes

$$R(a,b)\cdot\frac{w_{ab}(U)}{W} = \frac{w(U)}{W}$$

to the sum of voltages around C.

Case 3: a is in U_t and b is in U_s. Then the thicket U is also a ba-thicket, so U contributes $-w_{ab}(U)/W$ to $I(a,b)$. By Ohm's law and Lemma 4.1, U contributes

$$R(a,b)\cdot -\frac{w_{ab}(U)}{W} = -\frac{w(U)}{W}$$

to the sum of voltages around C.

Since C is a directed cycle, it contains as many arcs from U_s to U_t as arcs from U_t to U_s. It follows that an arbitrary thicket U contributes 0 to the sum of voltages around C, so the voltage law holds around C. □

The preceding theorem has the following useful specialization.

Corollary 4.4. *Let G be a connected graph representing an electrical network in which every edge has unit resistance. Assume the source s and the sink t are adjacent. Then the current in the edge st is*

$$I(s,t) = \frac{\tau(G) - \tau(G\backslash st)}{\tau(G)}, \tag{4.1}$$

where $G\backslash st$ is the graph obtained from G by deleting st.

Proof. Since every edge has resistance 1, every spanning tree has weight 1, giving $W = \tau(G)$. For any spanning tree T in G, the st-path in T contains the edge st if and only if T contains st. The spanning trees in G that contain st are exactly those spanning trees in G that are not spanning trees in $G\backslash st$, which gives $W(s,t) = \tau(G) - \tau(G\backslash st)$. Of course, an st-path cannot contain the edge ts, so $W(t,s) = 0$. Substituting all this into Theorem 4.3 yields

$$I(s,t) = \frac{W(s,t) - W(t,s)}{W} = \frac{\tau(G) - \tau(G\backslash st)}{\tau(G)}. \qquad \square$$

4.2 Effective resistance

In this section, we introduce the *effective resistance* between two vertices, first defined in Klein and Randić (1993). The content of this section serves as background for our results in Section 5.4.

Throughout this section, let G be a graph representing an electrical network in which every edge has unit resistance. We assume, as in the previous section, that each edge ab of G has a resistance $R(a,b)$. Moreover, we assume that no current flows in the network before we connect a battery.

Definition. *Let ab be an edge of G. Connect a battery's terminals to a and b. The* effective resistance *between a and b, denoted $r_{ab}(G)$, is the battery's voltage divided by the current supplied by the battery. We will write r_{ab} for $r_{ab}(G)$ when G is clear from context.*

Unlike the (intrinsic) resistance $R(a,b)$, the effective resistance r_{ab} is defined when a and b are not adjacent. Even when a and b are adjacent, r_{ab} in general differs from $R(a,b)$. For example, consider the graph H_1 in Figure 3.1. The edge between vertices 3 and 4 has resistance $R(3,4) = 1$ and effective resistance $r_{34} = 2/3$.

The effective resistance can be computed using standard techniques. These techniques involve replacing multiple resistors in parallel or in series with a single resistor. We do not provide further explanation; see Bollobás (1979) for an overview.

When the vertices a and b are adjacent, the effective resistance r_{ab} coincides in value with the current in Equation 4.1.

Proposition 4.5. *The effective resistance of every edge ab of G is*

$$r_{ab} = \frac{\tau(G) - \tau(G \backslash ab)}{\tau(G)} .$$

Proof. We assume a battery's terminals are connected to a and b, as in the definition of effective resistance. The effective resistance does not depend on the battery's particulars, so we can and will assume the battery supplies unit current. By Corollary 4.4, the current between a and b is

$$I(a, b) = \frac{\tau(G) - \tau(G \backslash ab)}{\tau(G)} .$$

Using Ohm's law, the battery's voltage is

$$V(a, b) = R(a, b) \cdot I(a, b) = \frac{\tau(G) - \tau(G \backslash ab)}{\tau(G)} ,$$

since every edge has unit resistance. Dividing this by the unit current supplied by the battery, we obtain the claimed value for the effective resistance r_{ab}. □

Stated in other words, the effective resistance of any edge is the proportion of spanning trees containing that edge.

One way to simplify an electrical network G is to *short* a set of vertices $S \subseteq V(G)$, that is, connect all the vertices in S with wires having zero resistance. Shorting S is equivalent to collapsing all the vertices in S into a single vertex. It is well-known that shorting a set of vertices cannot increase the effective resistance of any edge; see for example Doyle and Snell (1984: Section 2.2.2).

Short-cut principle. *Let $S \subseteq V(G)$, and let G' be the graph obtained from G by shorting S. Then $r_{ab}(G') \leq r_{ab}(G)$.*

Coppersmith et al. (1996: Proposition 2) use the short-cut principle to prove the following lower bound on effective resistance.

Proposition 4.6. *Let G be a connected graph representing an electrical network in which every edge has unit resistance. Then*

$$r_{ab} \geq \frac{1}{d_a + 1} + \frac{1}{d_b + 1}$$

for every edge ab of G.

This lower bound is improved by Palacios and Renom (2011), again by using the short-cut principle.

Proposition 4.7. *Let G be a connected graph representing an electrical network in which every edge has unit resistance. Then*

$$r_{ab} \geq \frac{d_a + d_b - 2}{d_a d_b - 1}.$$

for every edge ab of G.

4.3 Ferrers graphs

Ehrenborg and van Willigenburg (2004) define a class of bipartite graphs called *Ferrers graphs*. In this section, we will follow their proof that Conjecture 1.1 holds with equality for Ferrers graphs.

Definition. *A* Ferrers graph *is a bipartite graph with parts $X = \{x_0, \ldots, x_n\}$ and $Y = \{y_0, \ldots, y_m\}$ such that*

- *$x_0 y_m$ and $x_n y_0$ are edges, and*
- *whenever $x_p y_q$ is an edge, $x_i y_j$ is an edge for all $0 \leq i < p$ and $0 \leq j < q$.*

A Ferrers graph represents the integer partition $(d_{x_0}, \ldots, d_{x_n})$. Thus, there is a natural correspondence between Ferrers graphs and Ferrers diagrams, which also represent integer partitions. For example, the partition of 8 into $(4, 3, 1)$ is represented both by the Ferrers graph in Figure 4.1 and the Ferrers diagram in Figure 4.2.

We first prove a lemma about how adding an edge to a Ferrers graph changes the number of spanning trees.

Lemma 4.8. *Let H be a Ferrers graph without the edge $x_p y_q$, where $p, q \geq 1$. Suppose adding $x_p y_q$ to H forms another Ferrers graph G. Then*

$$\frac{\tau(G)}{\tau(H)} = \frac{(p+1)(q+1)}{pq}.$$

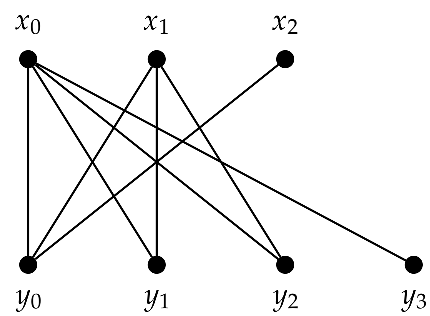

Figure 4.1 Ferrers graph representing the partition $(4, 3, 1)$.

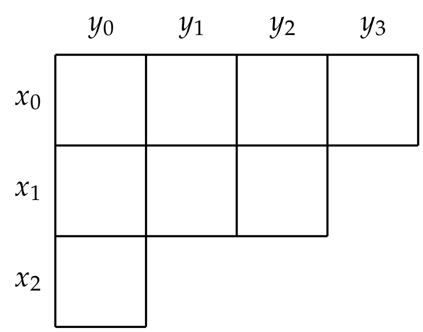

Figure 4.2 Ferrers diagram representing the partition $(4, 3, 1)$.

Proof. We view the Ferrers graph G as an electrical network. Assume a unit current enters at x_p and leaves at y_q. Write $N = (p+1)(q+1)$. To each edge $x_i y_j$, assign the resistance $R(x_i, y_j) = 1$ and the current

$$I(x_i, y_j) = \begin{cases} -1/N & \text{if } i < p,\ j < q, \\ p/N & \text{if } i = p,\ j < q, \\ q/N & \text{if } i < p,\ j = q, \\ (1+p+q)/N & \text{if } i = p,\ j = q, \\ 0 & \text{otherwise.} \end{cases}$$

We will check Kirchhoff's current law at x_p. We compute

$$\sum_{j=0}^{q} I(x_p, y_j) = q \cdot \frac{p}{N} + \frac{1+p+q}{N} = \frac{(p+1)(q+1)}{N} = 1,$$

so the current law holds at x_p. By a symmetric argument, the current law also holds at y_q. Next, we will check Kirchhoff's current law at x_i, for $i < p$.

We compute

$$\sum_{j=0}^{q} I(x_i, y_j) = q \cdot -\frac{1}{N} + \frac{q}{N} = 0,$$

so the current law holds at x_i. By a symmetric argument, the current law also holds at y_j, for $j < q$. No current flows into or out of x_i for $i > p$ or y_j for $j > q$, so the current law holds at every vertex.

Observe that the only nontrivial cycle in G is $x_p \to y_q \to x_i \to y_j \to x_p$, where $i < p$ and $j < q$. Thus, it suffices to check that Kirchhoff's voltage law holds around this cycle. Since every edge has resistance 1, Ohm's law gives $V(x_i, y_j) = R(x_i, y_j)$. The voltages around this cycle sum to

$$\frac{1+p+q}{N} - \frac{q}{N} - \frac{1}{N} - \frac{p}{N} = 0,$$

so the voltage law holds.

We assigned $I(x_p, y_q) = (1+p+q)/N$, where $N = (p+1)(q+1)$, so

$$I(x_p, y_q) = \frac{N - pq}{N} = 1 - \frac{pq}{(p+1)(q+1)}\,. \tag{4.2}$$

Corollary 4.4 gives the current between the source x_p and the sink y_q as

$$I(x_p, y_q) = \frac{\tau(G) - \tau(H)}{\tau(G)} = 1 - \frac{\tau(H)}{\tau(G)}\,.$$

Comparing this with Equation 4.2 yields the result. □

Conjecture 1.1 holds with equality for Ferrers graphs.

Theorem 4.9. *Let G be a Ferrers graph with parts X and Y. Then*

$$\tau(G) = \frac{1}{|X| \cdot |Y|} \prod_{v \in V(G)} d_v.$$

Proof. Label the vertices such that $X = \{x_0, \dots, x_n\}$ and $Y = \{y_0, \dots, y_m\}$. Fix n and m. We proceed by induction on $|E|$.

Let G_0 be the Ferrers graph whose only edges are $x_0 y_0$, $x_0 y_j$ for $1 \le j \le m$, and $x_i y_0$ for $1 \le i \le n$. Then G_0 has $n + m + 1$ edges, which is fewer than any other Ferrers graph with parts X and Y, so we take G_0 as the base case. Since G_0 is a tree, $\tau(G_0) = 1$. We find

$$\frac{1}{|X| \cdot |Y|} \prod_{v \in V(G_0)} d_v = \frac{d_{x_0} \cdot d_{y_0}}{(n+1)(m+1)} = 1 = \tau(G_0),$$

so the base case holds. If $n = 0$ or $m = 0$, then G_0 is the only Ferrers graph with parts X and Y, so we are done. Thus, assume $n, m \geq 1$.

Let G be a Ferrers graph with more edges than G_0. Remove an edge from G to form another Ferrers graph H (it can be shown that this is always possible); say the edge $x_p y_q$ is removed. Since H is a Ferrers graph, removing $x_p y_q$ decreased d_{x_p} from $p+1$ to p and decreased d_{y_q} from $q+1$ to q, so

$$\prod_{v \in V(G)} d_v = \frac{(p+1)(q+1)}{pq} \prod_{v \in V(H)} d_v.$$

By the induction hypothesis, the right-hand side product equals $\tau(H)$. Using Lemma 4.8,

$$\prod_{v \in V(G)} d_v = \frac{\tau(G)}{\tau(H)} \cdot \tau(H) = \tau(G),$$

completing the inductive step. □

4.4 Difference graphs

Chestnut and Fishkind (2013) study another class of bipartite graphs called *difference graphs*.

Definition. *A bipartite graph with parts X and Y is a* difference graph *if there exist a function $\phi : X \cup Y \to \mathbb{R}$ and a threshold $\alpha \in \mathbb{R}$ such that for all $x \in X$ and $y \in Y$, the graph has edge xy if and only if $\phi(x) + \phi(y) \geq \alpha$.*

They assert that Ehrenborg and van Willigenburg (2004) proved Theorem 4.9 for difference graphs, not for Ferrers graphs. This suggests that Ferrers graphs and difference graphs are equivalent, which is proven in Hammer et al. (1990: Theorem 2.3(4)). We give a more direct proof of this equivalence.

Theorem 4.10. *G is a Ferrers graph if and only if G is a connected difference graph.*

Proof. ($\Rightarrow$) Suppose G is a Ferrers graph with parts $X = \{x_0, \ldots, x_n\}$ and $Y = \{y_0, \ldots, y_m\}$. Then G is connected, since G contains the edges $x_0 y_0$, $x_0 y_j$ for $1 \leq j \leq m$, and $x_i y_0$ for $1 \leq i \leq n$.

To show that G is a difference graph, define the function $\phi : X \cup Y \to \mathbb{R}$ by $\phi(x_i) = n - i$ and $\phi(y_j) = d_{y_j}$. Since G is a Ferrers graph, the neighbors of

y_j are those x_i where $i \leq d_{y_j} - 1$, so we can establish the chain of equivalences

$$\begin{aligned} & \phi(x_i) + \phi(y_j) \geq n + 1 \\ \Leftrightarrow \quad & n - i + d_{y_j} \geq n + 1 \\ \Leftrightarrow \quad & i \leq d_{y_j} - 1 \\ \Leftrightarrow \quad & G \text{ has edge } x_i y_j. \end{aligned}$$

Thus, G is a difference graph with the function ϕ as defined and the threshold $\alpha = n + 1$.

($\Leftarrow$) Suppose G is a connected difference graph with parts X and Y, so there exists a function $\phi : X \cup Y \to \mathbb{R}$ and a threshold $\alpha \in \mathbb{R}$ as in the difference graph definition. Label the vertices in X as $x_0, \ldots, x_n$ and the vertices in Y as $y_0, \ldots, y_m$ such that $\phi(x_0) \geq \cdots \geq \phi(x_n)$ and $\phi(y_0) \geq \cdots \geq \phi(y_m)$. We must check the two conditions in the Ferrers graph definition.

First, we must show that $x_0 y_m$ and $x_n y_0$ are edges of G. Since G is connected, x_0 is adjacent to at least one y_j. Because $j = m$ minimizes $\phi(y_j)$, we have $\phi(x_0) + \phi(y_m) \geq \alpha$, so $x_0 y_m$ is an edge of G. A similar argument shows that $x_n y_0$ is also an edge of G.

Second, we must show that whenever $x_p y_q$ is an edge, $x_i y_j$ is an edge for all $0 \leq i < p$ and $0 \leq j < q$. Fix p, q, i, j so that $x_p y_q$ is an edge, $0 \leq i < p$, and $0 \leq j < q$. By our choice of labels, we have $\phi(x_i) \geq \phi(x_p)$ and $\phi(y_j) \geq \phi(y_q)$. It follows that

$$\phi(x_i) + \phi(y_j) \geq \phi(x_p) + \phi(y_q) \geq \alpha,$$

so $x_i y_j$ is an edge of G. □

Now Theorem 4.9, together with Theorem 4.10, shows that Conjecture 1.1 holds with equality for difference graphs. Hammer et al. (1990: Theorem 2.3) prove the following forbidden graph characterization of the bipartite graphs that are also difference graphs.

Proposition 4.11. *Let G be a bipartite graph. Then G is a difference graph if and only if G does not contain $K_2 \cup K_2$ as an induced subgraph.*

Thus, Conjecture 1.1 would follow in general if it were proved for bipartite graphs containing $K_2 \cup K_2$ as an induced subgraph.

Yet another common name for a difference graph is a *chain graph*; see Yannakakis (1981) for example.

Chapter 5

Results

This chapter contains original results of ours that represent progress towards Conjecture 1.1. For a summary of our results, refer to Section 1.3.

We will assume that all graphs are connected throughout this chapter, because if G were disconnected, then $\tau(G) = 0$ and the conjecture clearly holds.

5.1 Specific classes of graphs

We provide combinatorial proofs of Conjecture 1.1 for G having $|X| \leq 2$ and for G being an even cycle.

Proposition 5.1. *Let $G = (V, E)$ be a bipartite graph with parts X, Y, where $|X| = 1$. Then*

$$\tau(G) = \frac{1}{|Y|} \prod_{v \in V(G)} d_v.$$

Proof. Since G is connected, G must be the complete bipartite graph $K_{1,|Y|}$. The vertex in X has degree $|Y|$, and every other vertex has degree 1. Since G is itself a tree, $\tau(G) = 1$, showing the result. $\square$

Proposition 5.2. *Let $G = (V, E)$ be a bipartite graph with parts X, Y, where $|X| = 2$. Then*

$$\tau(G) \leq \frac{1}{2|Y|} \prod_{v \in V(G)} d_v.$$

Proof. Let $X = \{x_1, x_2\}$. For $i = 1, 2$, let Y_i denote the set of vertices in Y adjacent to only x_i in X. Let Y_{12} denote the set of vertices in Y adjacent to both x_1 and x_2.

Since G is connected, Y_{12} cannot be empty. Let T be a spanning tree in G. Then T must contain all the edges between X and Y_1 and all the edges between X and Y_2. For exactly one vertex $u \in Y_{12}$, the tree T contains the edges x_1u and x_2u. For every other vertex $w \in Y_{12}$, $w \neq u$, the tree T contains exactly one of the edges x_1w and x_2w.

There are $|Y_{12}|$ choices for u, and for each of the $|Y_{12}| - 1$ other vertices $w \in Y_{12}$, there are 2 choices for which of x_1w and x_2w is in T, so

$$\tau(G) = |Y_{12}| \cdot 2^{|Y_{12}|-1}.$$

We find

$$\begin{aligned}
2|Y| \cdot \tau(G) &= 2(|Y_1| + |Y_{12}| + |Y_2|) \cdot |Y_{12}| \cdot 2^{|Y_{12}|-1} \\
&= (|Y_1| + |Y_{12}| + |Y_2|) \cdot |Y_{12}| \cdot 2^{|Y_{12}|} \\
&= (|Y_1| + |Y_{12}|)(|Y_{12}| + |Y_2|) \cdot 2^{|Y_{12}|} - |Y_1| \cdot |Y_2| \cdot 2^{|Y_{12}|} \\
&\leq (|Y_1| + |Y_{12}|)(|Y_{12}| + |Y_2|) \cdot 2^{|Y_{12}|} \\
&= \prod_{v \in V(G)} d_v,
\end{aligned}$$

and then dividing through by $2|Y|$ gives the result. □

Conjecture 1.1 holds for all even cycles.

Proposition 5.3. *Conjecture* 1.1 *holds for even cycles.*

Proof. Let G be an even cycle with $2n$ vertices, where $n \geq 1$. The spanning trees in G are exactly the subgraphs obtained by deleting one edge in G. Since G has $2n$ edges, we have $\tau(G) = 2n$. Moreover $\prod_v d_v = 2^{2n}$ and $|X| \cdot |Y| = n^2$, so Conjecture 1.1 for G is equivalent to

$$2n^3 \leq 2^{2n}. \tag{5.1}$$

We proceed by induction on n. Equation 5.1 holds for $n \leq 2$, and we take $n = 2$ as the base case. As the inductive hypothesis, assume Equation 5.1 holds for some $k \geq 2$. Because $(k+1)/k \leq 3/2 < 2^{2/3}$, we have

$$2(k+1)^3 = \left(\frac{k+1}{k}\right)^3 \cdot 2k^3 \leq \left(2^{2/3}\right)^3 \cdot 2^{2k} = 2^{2(k+1)},$$

completing the inductive step. □

5.2 Connecting graphs by a new edge

We extend Proposition 1.3 by proving that Conjecture 1.1 holds under the operation of connecting two bipartite graphs by a new edge.

Proposition 5.4. *Let G and G' be bipartite graphs for which Conjecture* 1.1 *holds. Let X and Y be the parts of G, and let X' and Y' be the parts of G'. Choose vertices $x \in X$ and $x' \in X'$.*

Define the graph H by $V(H) = V(G) \cup V(G')$ and $E(H) = E(G) \cup E(G') \cup \{xx'\}$. Then the conjecture holds for H also.

Proof. Suppose that at least one of the parts X, Y, X', and Y' contains just one vertex. Then at least one of G or G' is a tree; without loss of generality, say G' is a tree. We may start from G and build up H by adding the vertex x' and the edge xx', and then adding the rest of G' similarly. The result follows by applying Proposition 1.3 at each step.

We may now assume that each of the parts X, Y, X', Y' contains at least two vertices. Then

$$\begin{aligned} |X| \cdot |Y'| &= (|X| - 1)(|Y'| - 1) + |X| + |Y'| - 1 \\ &\geq 1 \cdot 1 + |X| + |Y'| - 1 \\ &= |X| + |Y'| \\ &= |X \cup Y'|, \end{aligned}$$

and similarly $|X'| \cdot |Y| \geq |X' \cup Y|$. It follows that

$$|X| \cdot |Y| \cdot |X'| \cdot |Y'| \geq |X \cup Y'| \cdot |X' \cup Y|. \tag{5.2}$$

Every spanning tree in H is obtained by connecting a spanning tree in G and a spanning tree in G' by xx'. Since this correspondence is bijective,

$$\tau(H) = \tau(G) \cdot \tau(G'). \tag{5.3}$$

Since Conjecture 1.1 holds for G and G', we have

$$\begin{aligned} \tau(G) &\leq \frac{1}{|X| \cdot |Y|} \prod_{v \in V(G)} d_v(G), \\ \tau(G') &\leq \frac{1}{|X'| \cdot |Y'|} \prod_{v \in V(G')} d_v(G'), \end{aligned}$$

Multiplying these inequalities and using Equation 5.3,

$$\tau(H) \leq \frac{1}{|X| \cdot |Y| \cdot |X'| \cdot |Y'|} \left(\prod_{v \in V(G)} d_v(G) \right) \left(\prod_{v \in V(G')} d_v(G') \right).$$

Of course $d_v(H) \geq d_v(G)$ for every $v \in V(G)$, and in particular we have $d_x(H) = d_x(G) + 1$. This holds for the vertices of G' too, so

$$\begin{aligned} \tau(H) &< \frac{1}{|X| \cdot |Y| \cdot |X'| \cdot |Y'|} \left(\prod_{v \in V(G)} d_v(H) \right) \left(\prod_{v \in V(G')} d_v(H) \right) \\ &= \frac{1}{|X| \cdot |Y| \cdot |X'| \cdot |Y'|} \prod_{v \in V(H)} d_v(H). \end{aligned}$$

Observe that the parts of H are $X \cup Y'$ and $X' \cup Y$. By Inequality 5.2,

$$\tau(H) < \frac{1}{|X \cup Y'| \cdot |X' \cup Y|} \prod_{v \in V(H)} d_v(H),$$

showing that Conjecture 1.1 holds for H also. $\square$

5.3 Using spectral techniques

In this section, we approach Conjecture 1.1 by analyzing the normalized Laplacian spectrum. The following lemma is a direct consequence of Theorem 3.12 and Theorem 3.19.

Lemma 5.5. *Let G be a bipartite graph on $n \geq 3$ vertices. Then*

$$\tau(G) = \frac{\prod_v d_v}{|E(G)|} \cdot \prod_{i=1}^{n-2} \lambda_i.$$

Next, we follow a proof in Bozkurt (2012) of the following bound on the number of spanning trees in a bipartite graph.

Theorem 5.6. *Let G be a bipartite graph on $n \geq 2$ vertices. Then*

$$\tau(G) \leq \frac{\prod_v d_v}{|E(G)|}, \tag{5.4}$$

with equality if and only if G is complete bipartite.

Proof. If $n = 2$, then G is K_2 and both sides of Inequality 5.4 are 1, so the result holds. Assume $n \geq 3$. Using the AM-GM inequality on the statement of Lemma 5.5,

$$\tau(G) \leq \frac{\prod_v d_v}{|E(G)|}\left(\frac{\sum_{i=1}^{n-2} \lambda_i}{n-2}\right)^{n-2},$$

with equality exactly when the $n-2$ eigenvalues $\lambda_1, \ldots, \lambda_{n-2}$ are equal. By Theorem 3.13, this condition is equivalent to G being complete bipartite.

By Proposition 3.7, we have $\sum_{i=1}^{n-2} \lambda_i = n - \lambda_0 - \lambda_{n-1}$. In every graph, $\lambda_0 = 0$, and here $\lambda_{n-1} = 2$ (again by Theorem 3.12). Thus $\sum_{i=1}^{n-2} \lambda_i = n - 2$, and the result follows. □

Because there can be at most $|X| \cdot |Y|$ edges in a bipartite graph with parts X and Y, if Conjecture 1.1 were true, then Theorem 5.6 would follow. Specifically, the conjecture's result improves upon Theorem 5.6 by a factor of $|E(G)|/(|X| \cdot |Y|)$, which motivates the following definition.

Definition. *Let G be a bipartite graph with parts X and Y. The* bipartite density *of G, denoted $\rho(G)$, is the ratio $|E(G)|/(|X| \cdot |Y|)$.*

Equivalently, G contains $\rho(G)$ times as many edges as the complete bipartite graph $K_{|X|,|Y|}$. The bipartite density, along with the normalized Laplacian eigenvalues, is involved in the following reformulation of Conjecture 1.1.

Proposition 5.7. *Let G be a bipartite graph on $n \geq 3$ vertices with parts X and Y. Then Conjecture* 1.1 *holds for G if and only if*

$$\prod_{i=1}^{n-2} \lambda_i \leq \rho(G). \tag{5.5}$$

Proof. This follows directly from Lemma 5.5 and Conjecture 1.1. □

Our strategy to approach Conjecture 1.1 will be to use the following sufficient condition for the conjecture to hold.

Lemma 5.8. *Let G be a bipartite graph on $n \geq 3$ vertices with parts X and Y. Suppose, for some $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, we have*

$$\prod_{i=1}^{k} \lambda_i(2 - \lambda_i) \leq \rho(G).$$

Then Conjecture 1.1 *holds for G.*

Proof. We use the form of Conjecture 1.1 provided by Proposition 5.7. Proposition 3.14 gives $\lambda_{n-1-i} = 2 - \lambda_i$, so that

$$\prod_{i=1}^{k} \lambda_i(2-\lambda_i) = \left(\prod_{i=1}^{k} \lambda_i\right)\left(\prod_{i=1}^{k} \lambda_{n-1-i}\right) = \left(\prod_{i=1}^{k} \lambda_i\right)\left(\prod_{i=n-k-1}^{n-2} \lambda_i\right). \tag{5.6}$$

We distinguish three cases.

Case 1: n is even and $k = \lfloor (n-1)/2 \rfloor = n/2 - 1$. Then the right-hand side of Equation 5.6 involves each of the eigenvalues $\lambda_1, \ldots, \lambda_{n-2}$ once, so the supposition is equivalent to Inequality 5.5.

Case 2: n is odd and $k = \lfloor (n-1)/2 \rfloor = (n-1)/2$. Then the right-hand side of Equation 5.6 involves the middle eigenvalue $\lambda_{(n-1)/2}$ twice, and involves each of the other eigenvalues $\lambda_1, \ldots, \lambda_{n-2}$ once. By Proposition 3.14, we have $\lambda_{(n-1)/2} = 1$, so the supposition is again equivalent to Inequality 5.5.

Case 3: $k < \lfloor (n-1)/2 \rfloor$. In this case, there are some eigenvalues in the middle that are not involved in the right-hand side of Equation 5.6. These middle eigenvalues are the $n - 2k - 2$ eigenvalues $\lambda_{k+1}, \ldots, \lambda_{n-k-2}$. Their sum is, from Proposition 3.7 and Proposition 3.14,

$$\begin{aligned}
\sum_{i=k+1}^{n-k-2} \lambda_i &= \sum_{i=1}^{n-2} \lambda_i - \left(\sum_{i=1}^{k} \lambda_i + \sum_{i=n-k-1}^{n} \lambda_i\right) \\
&= n - \sum_{i=1}^{k} (\lambda_i + 2 - \lambda_i) \\
&= n - 2k.
\end{aligned}$$

Thus, applying the AM-GM inequality to these middle eigenvalues,

$$\prod_{i=k+1}^{n-k-2} \lambda_i \leq \left(\frac{\sum_{i=k+1}^{n-k-2} \lambda_i}{n-2k-2}\right)^{n-2k-2} = 1. \tag{5.7}$$

We finally show that Inequality 5.5 is satisfied, since

$$\begin{aligned}\prod_{i=1}^{n-2}\lambda_i &= \left(\prod_{i=1}^{k}\lambda_i\right)\left(\prod_{i=k+1}^{n-k-2}\lambda_i\right)\left(\prod_{i=n-k-1}^{n-2}\lambda_i\right)\\ &\le \left(\prod_{i=1}^{k}\lambda_i\right)\left(\prod_{i=n-k-1}^{n-2}\lambda_i\right)\\ &= \prod_{i=1}^{k}\lambda_i(2-\lambda_i)\\ &\le \rho(G),\end{aligned}$$

where we used Equation 5.7, Equation 5.6, and the supposition in that order. □

The preceding lemma gives a way Conjecture 1.1 by obtaining upper bounds on the first k eigenvalues. Because the spectrum is symmetric (Proposition 3.14), these would also be lower bounds on the last k eigenvalues. In this report, we focus on the smallest nontrivial eigenvalue λ_1. Specializing the preceding lemma to $k = 1$ gives the following lemma.

Lemma 5.9. *Let G be a bipartite graph on $n \ge 3$ vertices with parts X and Y. Suppose*

$$\lambda_1(2-\lambda_1) \le \rho(G).$$

Then Conjecture 1.1 *holds for G.*

For graphs having a cut vertex of degree 2, we have the following upper bound on λ_1.

Proposition 5.10. *Let G be a graph having a cut vertex x of degree* 2. *Write $m = |E(G)|$. Then*

$$\lambda_1 \le \frac{2m}{4m-5+\sqrt{8m^2-28m+25}}.$$

Proof. Let G_1 and G_2 be the two components obtained from G by removing the cut vertex x. Write $\alpha = \operatorname{vol} V(G_1)$ and $\beta = \operatorname{vol} V(G_2)$, where $\operatorname{vol} V(G_i) = \sum_{v\in V(G_i)} d_v(G)$. Then, define $\mathbf{f} \in \mathbb{R}^n$ by

$$f_v = \begin{cases} c_1 & \text{if } v \in V(G_1),\\ c_2 & \text{if } v \in V(G_2),\\ 0 & \text{if } v = x.\end{cases}$$

where the c_i's are constants that depend only on α and β. We will specify the c_i's later. Observe that every edge of G connects two vertices in G_1, connects two vertices in G_2, or is incident to x. Thus,

$$\sum_{i \sim j} (f_i - f_j)^2 = c_1^2 + c_2^2.$$

Also,

$$\sum_{i,j} (f_i - f_j)^2 d_i d_j = 2c_1^2 \alpha + 2c_2^2 \beta + (c_1 - c_2)^2 \alpha\beta,$$

where each term corresponds to a choice of two parts from the partition $\{V(G_1), V(G_2), \{x\}\}$ of the vertex set $V(G)$. Substituting this sum and the previous one into Proposition 3.16, we obtain the upper bound

$$\lambda_1 \leq \operatorname{vol} G \cdot \frac{c_1^2 + c_2^2}{2c_1^2 \alpha + 2c_2^2 \beta + (c_1 - c_2)^2 \alpha\beta}. \tag{5.8}$$

We choose

$$\begin{aligned} c_1 &= \frac{\sqrt{(\alpha - \beta)^2 + (\alpha\beta)^2} - \alpha\beta}{\alpha - \beta} - 1, \\ c_2 &= \frac{\sqrt{(\alpha - \beta)^2 + (\alpha\beta)^2} - \alpha\beta}{\alpha - \beta} + 1, \end{aligned}$$

since computations with Mathematica indicate that these choices of the c_i's minimize the upper bound in Equation 5.8. Choosing these c_i's gives the upper bound

$$\lambda_1 \leq \frac{\operatorname{vol} G}{\alpha + \beta + \alpha\beta + \sqrt{(\alpha - \beta)^2 + (\alpha\beta)^2}}. \tag{5.9}$$

Because x is adjacent to some vertex in G_1 and to some vertex in G_2, we have $\alpha, \beta \geq 1$. The only other constraint on α, β is that

$$\alpha + \beta = \operatorname{vol} G - \operatorname{vol}\{x\} = \operatorname{vol} G - 2.$$

It can be shown that under these constraints, the upper bound in Equation 5.9 is worst when $\alpha = 1$ and $\beta = 2m - 3$ or vice versa. The result then follows by substituting these values for α, β into Equation 5.9 and using the fact that $\operatorname{vol} G = 2m$. $\square$

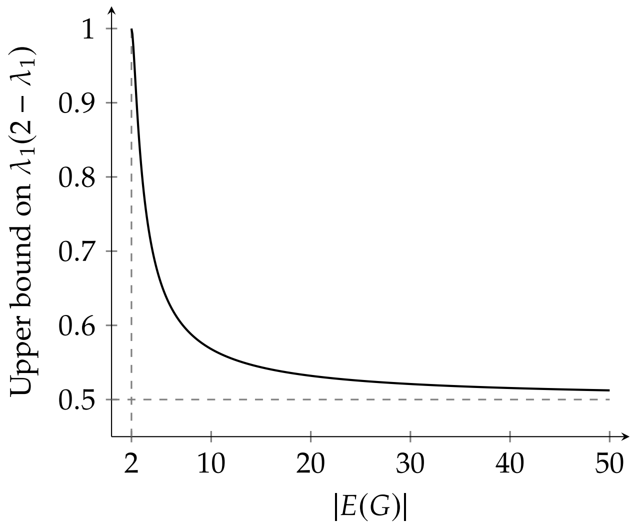

Figure 5.1 Upper bound on $\lambda_1(2-\lambda_1)$ for $2 \le |E(G)| \le 50$.

Figure 5.1 shows the upper bound on $\lambda_1(2-\lambda_1)$ given by Proposition 5.10. This upper bound is decreasing in $|E(G)|$ and approaches 0.5 from above as $|E(G)| \to \infty$.

We establish as a lemma that for graphs with few edges, Conjecture 1.1 is settled by the results of Garrett and Klee (2014). This lemma will let us give a better upper bound on $\lambda_1(2-\lambda_1)$, since this bound gets better as $|E(G)|$ increases (see Figure 5.1).

Lemma 5.11. *Conjecture* 1.1 *holds for bipartite graphs with at most* 12 *edges.*

Proof. Let G be a bipartite graph. If G has at most 11 vertices, then the conjecture holds by Theorem 1.2. Thus, we may assume G has at least 12 vertices, so it has at least 11 edges. It remains to check two cases:

Case 1: G has 11 edges and 12 vertices, or G has 12 edges and 13 vertices. Then G is a tree, so the conjecture holds for G by Corollary 1.4.

Case 2: G has 12 edges and 12 vertices. Then G can be obtained from an even cycle by the operation in Proposition 1.3. Since the conjecture holds for even cycles by Proposition 5.3, the conjecture holds for G also. □

Our main result proves Conjecture 1.1 for sufficiently dense graphs containing a cut vertex of degree 2.

Theorem 5.12. *Let G be a bipartite graph. Suppose that $\rho(G) \ge 0.544$ and that G contains a cut vertex x of degree* 2. *Then Conjecture* 1.1 *holds for G.*

Proof. Let G_1 and G_2 be the two components obtained from G by removing the cut vertex x. We check two cases:

Case 1: G has at most 14 edges. Then each of G_1 and G_2 has at most 12 edges, so by Lemma 5.11, the conjecture holds for each G_i. Since we can reconstruct G from the G_i's using the operations in Proposition 1.3 and Proposition 5.4, the conjecture holds for G also.

Case 2: G has at least 15 edges. The upper bound on λ_1 given by Proposition 5.10 is $\lambda_1 \leq 30/(55 + \sqrt{1405}) \approx 0.324$. This gives

$$\lambda_1(2 - \lambda_1) < 0.544 \leq \rho(G),$$

which is sufficient to prove the conjecture by Lemma 5.9. □

The preceding result proves Conjecture 1.1 for graphs that are globally dense (have high $\rho(G)$) but also locally sparse (contain a cut vertex). These conditions suggest that the deviation from equality in the conjecture does not depend on the graph's density in a straightforward manner. This observation is consistent with equality for Ferrers graphs (Theorem 4.9), because Ferrers graphs can range in density from trees to complete bipartite graphs.

5.4 Using electrical network analysis

Let G be a bipartite graph. We also adopt the setup from Section 4.2, so we view G as representing an electrical network in which every edge has unit resistance.

We are interested in a lower bound for the effective resistance r_{xy} in terms of the degrees d_x and d_y. Recall as examples Proposition 4.6 and Proposition 4.7, which apply to graphs in general. We establish a lower bound for bipartite graphs that, in some cases, improves on the mentioned bounds.

The following technical lemma will be helpful.

Lemma 5.13. *Let G be a bipartite graph with parts X and Y. Then $|E(G)| \geq d_x + d_y$ for every edge xy of G, assuming that $G \backslash xy$ is connected.*

Proof. We know $G\backslash xy$ is connected and does not contain the edge xy, so it contains $d_x - 1$ edges incident to x, another $d_y - 1$ edges incident to y, and at least one more edge. We have identified $d_x + d_y - 1$ edges in $G\backslash xy$, so adding xy back gives

$$|E(G)| \geq d_x + d_y.$$ □

Proposition 5.14. *Let G be a bipartite graph with parts X and Y. View G as representing an electrical network in which every edge has unit resistance. Then*

$$r_{xy} \geq \left(1 + \left(\frac{1}{d_x - 1} + \frac{1}{d_y - 1} + \frac{1}{|E(G)| - d_x - d_y + 1}\right)^{-1}\right)^{-1}$$

for every edge xy of G, assuming that $G \backslash xy$ is connected.

Proof. Write $m = |E(G)|$. Fix an edge xy of G, assuming $x \in X$ and $y \in Y$. Short $X \backslash \{x\}$ by collapsing it into a vertex x_1, and short $Y \backslash \{y\}$ by collapsing it into a vertex y_1. Let G' denote the graph obtained, so $V(G) = \{x, y, x_1, y_1\}$.

In the graph G', the number of edges between x and y is 1, that between x and y_1 is $d_x - 1$, and that between y and x_1 is $d_y - 1$. Since G' is bipartite, all of the $m - d_x - d_y + 1$ remaining edges are between x_1 and y_1, and there is at least one remaining edge by Lemma 5.13.

We can thus replace the edges between x and y_1 by an edge with resistance $1/(d_x - 1)$, replace the edges between y and x_1 by an edge with resistance $1/(d_y - 1)$, and replace the edges between x_1 and y_1 by an edge with resistance $1/(m - d_x - d_y + 1)$. The result follows from directly computing the effective resistance in the shorted network, and then applying the short-cut principle. □

Using this bound, we give a sufficient condition for the ratio $\tau(G)/\prod_v d_v$ to not increase when an edge xy is deleted from G.

Theorem 5.15. *Let G be a bipartite graph with parts X and Y. Suppose xy is an edge of G such that $d_x(G) \cdot d_y(G) \geq |E(G)|$, and assume that $G \backslash xy$ is connected. Then*

$$\frac{\tau(G \backslash xy)}{\prod_v d_v(G \backslash xy)} \leq \frac{\tau(G)}{\prod_v d_v(G)} .$$

Proof. Write $m = |E(G)|$. For convenience, let

$$b = \left(1 + \left(\frac{1}{d_x - 1} + \frac{1}{d_y - 1} + \frac{1}{m - d_x - d_y + 1}\right)^{-1}\right)^{-1}$$

denote the lower bound in Proposition 5.14, so from Proposition 5.14, we have $r_{xy} \geq b$. By Proposition 4.5,

$$\frac{\tau(G) - \tau(G \backslash xy)}{\tau(G)} \geq b,$$

which rearranges to

$$\frac{\tau(G\backslash xy)}{\tau(G)} \leq 1 - b.$$

The result would follow from

$$1 - b \leq \frac{\prod_v d_v(G\backslash xy)}{\prod_v d_v(G)} = \frac{(d_x - 1)(d_y - 1)}{d_x d_y}, \tag{5.10}$$

which we now verify. We compute

$$\frac{(d_x - 1)(d_y - 1)}{d_x d_y} - 1 + b = \frac{(d_x - 1)(d_y - 1)(d_x d_y - m)}{d_x d_y \left(d_x d_y (m - d_x - d_y + 2) - m\right)}$$

Here the numerator is nonnegative because we supposed that $d_x d_y \geq m$. The denominator is positive because

$$d_x d_y (m - d_x - d_y + 2) - m = d_x d_y (m - d_x - d_y + 1) + (d_x d_y - m),$$

in which $m - d_x - d_y + 1 > 0$ by Lemma 5.13 and $d_x d_y \geq m$ by the supposition. It follows that

$$\frac{(d_x - 1)(d_y - 1)}{d_x d_y} - 1 + b \geq 0,$$

which verifies Inequality 5.10 and completes the proof. □

The following corollary gives a degree condition such that the conjecture holds under the operation of removing an edge.

Corollary 5.16. *Let G be a bipartite graph for which Conjecture* 1.1 *holds. Suppose xy is an edge of G such that $d_x(G) \cdot d_y(G) \geq |E(G)|$. Then the conjecture holds for $G\backslash xy$ also.*

Proof. If $G\backslash xy$ is connected, then the result directly follows from Theorem 5.15 and Conjecture 1.1. Otherwise, the result still holds because Conjecture 1.1 holds for disconnected graphs. □

Here is an example of how the preceding corollary can be used to show that Conjecture 1.1 holds for certain bipartite graphs.

Proposition 5.17. *Let K' be the graph obtained by removing a matching from the complete bipartite graph $K_{|X|,|Y|}$. Then Conjecture* 1.1 *holds for K'.*

Proof. Write $K = K_{|X|,|Y|}$. Since K is a Ferrers graph, Conjecture 1.1 holds for K by Theorem 4.9. For all $x \in X$ and $y \in Y$, we have $d_x(K) \cdot d_y(K) = |Y| \cdot |X| = |E(K)|$.

Remove the edges of the matching one by one. Right before each edge $x_i y_i$ is removed from the intermediate graph K_i, we still have $d_{x_i}(K_i) \cdot d_{y_i}(K_i) = |E(K)|$, because no edge incident to x_i or y_i has been removed before. Thus, Corollary 5.16 applies when each edge is removed, which gives the result. □

This result establishes Conjecture 1.1 for a new class of graphs. Indeed, whenever the matching removed contains more than one edge, Proposition 4.11 implies that K' cannot be a Ferrers graph.

Chapter 6

Conclusion

6.1 Discussion

In this report, we approached Ehrenborg's conjecture (Conjecture 1.1) along three different routes: combinatorial proofs, spectral techniques, and electrical network analysis. We discuss each route in turn.

6.1.1 Combinatorial proofs

In Section 5.1, we gave combinatorial proofs for G having $|X| \leq 2$ and for G being an even cycle. These proofs rely on how straightforward it is to count spanning trees in such graphs. For graphs in general, Kirchhoff's theorem (Theorem 3.18) reduces the problem of counting spanning trees to that of computing a determinant. However, except for special classes of graphs, it is rare that the number of spanning trees is easy to compare to the product of vertex degrees, as is required to resolve Conjecture 1.1. That said, it would be interesting to see combinatorial proofs for other specific classes of bipartite graphs.

We gave another combinatorial proof in Section 5.2. When two bipartite graphs G and G' are connected by an edge to form a graph H, there is the simple relation $\tau(H) = \tau(G) \cdot \tau(G')$. This relation let us extend a result of Garrett and Klee (2014) by proving that Ehrenborg's conjecture holds under the operation of connecting two graphs by a new edge.

6.1.2 Spectral techniques

We found a sufficient condition for Ehrenborg's conjecture in Section 5.3 using techniques from spectral graph theory. This condition involved the smallest nontrivial eigenvalue λ_1 and the bipartite density $\rho(G)$. In order to exploit a variational characterization due to Chung (1997), we supposed that the graph G contains a cut vertex of degree 2. This vertex served as a bottleneck, letting us obtain a good upper bound on λ_1 and eventually prove Conjecture 1.1 when $\rho(G)$ is sufficiently large.

As Lemma 5.8, we state a sufficient condition for Ehrenborg's conjecture that involves the k smallest eigenvalues. However, as Butler (2008: Section 5.5) points out, the middle eigenvalues $\lambda_2, \ldots, \lambda_{n-3}$ are much less well-understood than λ_1 and λ_{n-1}. In particular, suppose we write down a variational characterization of λ_2, in analogy with the characterizations of λ_1 (Propositions 3.15 and 3.16). But those characterizations of λ_1 were easy to work with because they involved the eigenvector $D^{1/2}\mathbb{1}$ corresponding to λ_0, while an analogous characterization of λ_2 would need to depend on the eigenvector corresponding to λ_1.

There are ways to control all the eigenvalues at once. One example is the technique of *eigenvalue interlacing*, surveyed in Haemers (1995). However, because the number of spanning trees involves the product of all $n-1$ nontrivial eigenvalues of L (Theorem 3.18) or $\mathcal{L}$ (Theorem 3.19), even a small difference between a bound and the actual eigenvalue can render the bound too weak to establish Conjecture 1.1.

6.1.3 Electrical network analysis

Section 5.4 saw us apply the theory of electrical networks towards Conjecture 1.1. Using the short-cut principle, we obtained a lower bound on the effective resistance of each edge, which is related via Proposition 4.5 to the proportion of spanning trees containing that edge.

Unfortunately, even with our lower bound, Theorem 5.15 requires the strict degree condition $d_x(G) \cdot d_y(G) \geq |E(G)|$ in order to guarantee that removing the edge xy does not increase the ratio $\tau(G)/\prod_v d_v$. One way forward would be to improve our lower bound.

It might be fruitful to investigate a formula, derived in Ghosh et al. (2008: Section 2.4), for the effective resistance in terms of the Moore–Penrose pseudoinverse $\mathcal{L}^+$ of $\mathcal{L}$. Relatedly, the pseudoinverse $\mathcal{L}^+$ has a neat expression when the graph is bipartite (van Dooren and Ho, 2005: Theorem 1).

6.2 Future work

6.2.1 Cartesian product

Proposition 5.4 is an example of the strategy of showing the conjecture holds under operations that combine two bipartite graphs. Another such operation is the *Cartesian product* of two graphs G and G', which produces a new graph denoted $G \square G'$. The Cartesian product seems promising because it enjoys the following properties:

(i) (Sabidussi, 1957: Lemma 2.6) The Cartesian product $G \square G'$ is bipartite if and only if G and G' are bipartite.

(ii) (Sabidussi, 1960: Corollary 2.15) Every connected graph G has a factorization with respect to the Cartesian product into *prime factors*, which are graphs that cannot be expressed as a nontrivial Cartesian product. Moreover, this factorization is unique up to isomorphisms.

(iii) (Fiedler, 1973: Property 3.4) Let G be a graph on n vertices and let G' be a graph on m vertices. Including multiplicities, the nm eigenvalues of $L(G \square G')$ are the values of $\mu_i(G) \cdot \mu_j(G')$ for $0 \leq i < n$ and $0 \leq j < m$.

For example, Wu and Chung (2014) apply the eigenvalue relation in property (iii) to count spanning trees in certain regular graphs.

6.2.2 Laplacian eigenvalues

The proof of Theorem 5.12 uses an upper bound on the eigenvalue λ_1 of $\mathcal{L}$. Order the vertices by their degrees so that $d_1 \leq d_2 \leq \cdots \leq d_n$. It can be shown using Theorem 3.18 that Conjecture 1.1 is equivalent to

$$\prod_{i=1}^{n-1} \mu_i \leq \frac{n}{|X| \cdot |Y|} \prod_{i=1}^{n} d_i. \tag{6.1}$$

This suggests approaching the conjecture by finding an upper bound on the eigenvalues μ_i of L.

Based on a result of Brouwer and Haemers (2008), it is shown by Farber and Kaminer (2011) that for $1 \leq i \leq n-1$, we have $\mu_i \leq d_i + i - 1$ except for one graph. This gives

$$\prod_{i=1}^{n-1} \mu_i \leq \prod_{i=1}^{n-1} (d_i + i - 1). \tag{6.2}$$

Checking some bipartite graphs reveals that the upper bound in Inequality 6.2 can be many orders of magnitude larger than that in Inequality 6.1. Perhaps one can improve the result of Brouwer and Haemers (2008) for the case of bipartite graphs.

6.2.3 Local graph operations

We can rewrite Conjecture 1.1 as

$$\frac{\tau(G)}{\prod_v d_v(G)} \leq \frac{1}{|X| \cdot |Y|}\,.$$

Since the conjecture holds with equality for Ferrers graphs (Theorem 4.9), it can be reformulated as follows.

Conjecture 6.1. *Let G be a connected bipartite graph with parts X and Y. Then*

$$\frac{\tau(G)}{\prod_v d_v(G)} \leq \frac{\tau(G_F)}{\prod_v d_v(G_F)}$$

for any Ferrers graph G_F with parts X and Y.

Fix parts X and Y, and let $\mathcal{B}$ denote the set of all connected bipartite graphs with parts X and Y. The reformulation in Conjecture 6.1 suggests finding a graph operation whose effect on $\tau(G)/\prod_v d_v(G)$ we understand well. We now state the properties that we want such a graph operation to have.

Proposition 6.2. *Conjecture* 6.1 *would be implied by the existence of a map $f : \mathcal{B} \to \mathcal{B}$ with the following properties*:

(*i*) *applying f to any $G \in \mathcal{B}$ does not decrease $\tau(G)/\prod_v d_v(G)$, and*

(*ii*) *starting from any $G \in \mathcal{B}$, one can obtain a Ferrers graph by applying f zero or more times.*

The operation of adding an edge certainly satisfies property (ii), so one might hope that it satisfies property (i) as well. For some evidence in this direction, consider what happens when we repeat the proof of Theorem 5.15 using the r_{xy} bound from Proposition 4.7 instead of our bound from Proposition 5.14.

Proposition 6.3. *Let G be a connected bipartite graph with parts X and Y. Suppose xy is an edge of G. Then*

$$\frac{\tau(G\backslash xy)}{(d_x-1)(d_y-1)} \le \frac{\tau(G)}{d_xd_y-1}, \tag{6.3}$$

where we assume that $G\backslash xy$ is connected.

Proof. From Proposition 4.7, we have

$$r_{xy} \ge \frac{d_x+d_y-2}{d_xd_y-1}.$$

Using Proposition 4.5,

$$\frac{\tau(G\backslash xy)}{\tau(G)} = 1 - r_{xy} \le 1 - \frac{d_x+d_y-2}{d_xd_y-1} = \frac{(d_x-1)(d_y-1)}{d_xd_y-1}. \qquad \square$$

Imagine if Proposition 6.3 held with the small change that the right-hand side of Inequality 6.3 is $\tau(G)/(d_xd_y)$ instead of $\tau(G)/(d_xd_y-1)$. Then the operation of adding an edge would indeed satisfy property (i) of Proposition 6.2, establishing Conjecture 1.1. However, adding an edge does not satisfy property (i). As a counterexample, consider the graph H_2 in Figure 6.1.

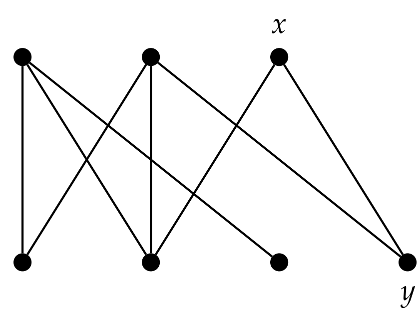

Figure 6.1 The counterexample graph H_2.

We compute

$$\frac{\tau(H_2\backslash xy)}{\prod_v d_v(H_2\backslash xy)} = \frac{4}{54} > \frac{15}{216} = \frac{\tau(H_2)}{\prod_v d_v(H_2)},$$

verifying that H_2 is a counterexample. Note that neither H_2 nor $H_2\backslash xy$ are Ferrers graphs.

Having seen that adding an edge does not satisfy Proposition 6.2, we define two graph operations that might.

Definition. *Let $G \in \mathcal{B}$. Suppose G is not complete bipartite, so there exists a vertex $x \in X$ whose neighborhood is not all of Y. Define the* spread graph, *denoted* spread(G, x), *as the graph obtained from G by connecting x to every vertex in Y.*

Definition. *Let $G \in \mathcal{B}$. Suppose G is not a Ferrers graph, so G contains $K_2 \cup K_2$ as an induced subgraph by Proposition* 4.11. *Equivalently, there exist vertices $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $x_1 \sim x_2$ and $y_1 \sim y_2$, but $x_1 \nsim y_2$ and $x_2 \nsim y_1$. Define the* cross graph, *denoted* cross(G, x_1y_1, x_2y_2), *as the graph obtained from G by adding the edges x_1y_2 and x_2y_1.*

Starting from any connected bipartite graph G, repeatedly applying the spread operation will result in a complete bipartite graph (which is a Ferrers graph). Similarly, starting from G and repeatedly applying the cross operation will result in a Ferrers graph. Thus, both of these operations have property (ii) in Proposition 6.2. We conjecture that these operations have property (i) as well.

Conjecture 6.4. *Let G be a connected bipartite graph with parts X and Y. If G' is a graph obtained from G via a spread operation, then*

$$\frac{\tau(G)}{\prod_v d_v(G)} \leq \frac{\tau(G')}{\prod_v d_v(G')}.$$

Conjecture 6.5. *Let G be a connected bipartite graph with parts X and Y. If G' is a graph obtained from G via a cross operation, then*

$$\frac{\tau(G)}{\prod_v d_v(G)} \leq \frac{\tau(G')}{\prod_v d_v(G')}.$$

We have obtained empirical data in support of Conjectures 6.4 and 6.5 by randomly generating bipartite graphs using the Python package NetworkX (Hagberg et al., 2008). Because neither of our conjectures directly implies the other, we believe our conjectures indicate two distinct approaches towards confirming Conjecture 1.1 in general.

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