An isomorphism between two $R$-modules $M$ and $N$ is a group homomorphism that also respects the module structure. That is, it is a function $f : M \to N$ such that:

1. $f(m + m') = f(m) + f(m')$ for all $m, n \in M$.
2. $f(r \cdot m) = r \cdot f(m)$ for all $m \in M$ and $r \in R$.

If $f$ is a homomorphism that is also a bijection, then we say that $f$ is an isomorphism.

Example. Let $V$ and $W$ be real vector spaces - i.e. modules over $\mathbb{R}$. Then a homomorphism between $V$ and $W$ is a function $f$ such that for all $v, v' \in V$ and $a \in \mathbb{R}$:

1. $f(v + v') = f(v) + f(v')$.
2. $f(av) = af(v)$.

That is, $f$ is a linear transformation between the vector spaces. If $f$ happens to be an invertible linear transformation between vector spaces of the same dimension, then it is a bijection, and thus an isomorphism.

Example. For $\mathbb{Z}$-modules, abelian group homomorphisms are automatically $\mathbb{Z}$-module homomorphisms. The first condition is exactly the homomorphism condition, and if $f : M \to N$ is a homomorphism of abelian groups, then $f(k \cdot m) = f(m + \ldots + m) = f(m) + f(m) + \ldots + f(m) = k \cdot f(m)$ for any $k \in \mathbb{Z}$ and $m \in M$.

Example. Both $\mathbb{C}$ and $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$ are $\mathbb{R}$-modules. They are, in fact, isomorphic as modules.

Define $f : \mathbb{C} \to \mathbb{R}^2$ by $f(x + iy) = (x, y)$. It is clear this map is a bijection. Then we check $f((x + yi) + (t + wi)) = f((x + t) + (y + w)i) = (x + t, y + w) = (x, y) + (t, w) = f(x + yi) + f(t + wi)$. Similarly, $f(a \cdot (x + yi)) = f(ax + ayi) = (ax, ay) = a \cdot (x, y) = a \cdot f(x + yi)$. Of course, I can make these two sets into rings in obvious ways, but they are NOT isomorphic as rings. Just as modules.

**Theorem 1.** Let $f : M \to N$ be a module homomorphism. Then:

1. $\text{Ker}(f) = \{m \in M \mid f(m) = 0_n\}$ is a submodule of $M$.
2. $\text{Im}(f) = \{n \in N \mid n = f(m) \text{ for some } m \in M\}$ is a submodule of $N$.

Naturally, once we have kernels and things floating around, we want to start thinking about quotients.

So let $N \leq M$ be a submodule of an $R$-module $M$. Then the cosets of $N$ in $M$ are the sets

$$m + N = \{m + n \mid m \in M, n \in N\}.$$ 

**Theorem 2.** For any submodule $N \leq M$, the set of cosets $\{m + N\}$ is an $R$ module with $R$-action

$$r \cdot (m + N) = (r \cdot m) + N.$$ 

Proof. Since $M$ is an abelian group and $N$ is a subgroup, we know that $M/N = \{m + N \mid m \in M\}$ is a group. So we just have to define an $R$-module structure on the group. We want to define

$$r \cdot (m + N) = (r \cdot m) + N,$$

but as always, I have to worry that it’s well defined.

So check. If $m' \in m + N$, then $m' = m + n$ for some $n \in N$. Then $r \cdot (m' + N) = r \cdot (m + n) + N = r \cdot m + r \cdot n + N = r \cdot m + N$. So the module structure is well defined. Then checking the axioms is easy:

$$r \cdot (s \cdot (m + N)) = r \cdot (s \cdot m + N) = (r \cdot s \cdot m) + N = rs \cdot m + N = rs \cdot (m + N),$$ etc.
Example.

Consider the polynomial ring \( R = \mathbb{R}[x] \). Let \( V \) be a 3-dimensional vector space, and let \( R \) act on it by letting \( x \) act by the matrix \[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]. Then notice that \( W = \langle v_1 \rangle \) is a submodule of \( V \), since \( x \cdot v_1 = v_1 \). So we should be able to think about the quotient \( V/W \). It should also be a vector space, and in fact it has basis \( v_2 + W \) and \( v_3 + W \). How does \( R \) act on it? Well, I just need to know how \( x \) acts on it. But \( x \cdot (v_2 + W) = x \cdot v_2 + W = v_1 + v_2 + W = v_2 + W \), and \( x \cdot (v_3 + W) = v_2 + v_3 + W \). So \( x \) acts by the matrix \[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]. We get our quotient module by basically taking the matrix that described the original module and erasing the part that came from what we quotiented out by.

So, we have quotient modules. Again, we ask about the isomorphism theorems. Again, yes.

Ok, so modules are great, but what do we use them for? Well, the idea is to do the same type of thing we did with group actions - the structure of the ring is linked to the structure of the module. And if we know a lot about one, hopefully it will give us information about the other.

So here’s an example of that.

**Theorem 3.** A module \( M \) over \( \mathbb{R}[x] \) is described by the matrix that defines how \( x \) acts. Two modules given by the matrices \( A \) and \( B \) are isomorphic if and only if \( A \) and \( B \) are similar matrices. I.e. \( A = P^{-1}BP \) for some invertible matrix \( P \).

Pf. Let \( V \) and \( W \) be modules over \( \mathbb{R}[x] \), so that the action of \( x \) on \( V \) is given by \( A \), and the action of \( x \) on \( N \) is given by \( B \). Since modules can only be isomorphic if they have the same dimension, we may as well assume that \( A \) and \( B \) are the same size.

First, assume that \( A = P^{-1}BP \) for some \( P \). This means that \( PA = BP \). We want to define an \( R \)-module isomorphism from \( V \) to \( W \). But an isomorphism of vector spaces is just an invertible linear transformation. And an \( R \)-module homomorphism is a map \( f \) satisfying \( f(r \cdot v) = r \cdot f(v) \) for any \( v \in V \) and \( r \in R \). But of course this works out for any scalar, since \( f \) is linear. So we just need to check that this is true for \( x \).

So define \( f : V \to W \) by \( f(v) = P \). Then we compute:

\[
f(x \cdot v) = f(Av) = PAv = BPv = x \cdot Pv = x \cdot f(v).
\]

So this is an \( R \)-module isomorphism.

Next, assume that \( M \) and \( N \) are isomorphic. Then there is an isomorphism between \( M \) and \( N \) as modules, which must be an isomorphism of vector spaces. That means the isomorphism is given by some matrix \( P \) that sends \( M \) to \( N \). Then the condition that \( f(x \cdot v) = x \cdot f(v) \) becomes exactly the condition \( PAv = BPv \), so \( A = P^{-1}BP \), and \( A \) and \( B \) are similar. □

So if we want to understand when matrices are similar, we should try and understand modules over \( \mathbb{R}[x] \).

In fact, the general goal is to find a nice classification theorem for modules over any PID. This will tell us about \( \mathbb{R}[x] \) (and therefore similar matrices), and also about \( \mathbb{Z} \) (and therefore abelian groups).