Today I want to take a look at modules over the ring $R = \mathbb{C}[x]$. As a quick reminder of what we know about modules over polynomial rings:

Fact: A $\mathbb{C}[x]$-module is a vector space $V$, and the action of $R$ on $V$ is completely described by choosing a matrix $A$ and defining that $x$ acts on $V$ according to the matrix $A$.

Fact: Two $\mathbb{C}[x]$-modules are isomorphic if and only if their corresponding matrices are similar, i.e. $A = P^{-1}BP$ for some invertible matrix $P$.

Combining these facts with the theorem about modules over a PID gives us a strong result about matrices.

Let’s start from the module theorem.

**Theorem 1.** Every finitely-generated $R$-module is isomorphic to one of the form $R^n \times R/(p_1^{b_1}) \times R/(p_2^{b_2}) \times \ldots \times R/(p_m^{b_m})$ for some prime elements $p_i$ of $R$. (Some of the $p_i$’s might repeat.)

Step 1: What are the primes in $R$?

Well, none of the units are prime in $R$, so we just look at polynomials of positive degree. Let $f(x)$ be any polynomial of degree $n$. Then $f$ has exactly $n$ roots in the complex numbers, $z_1, z_2, \ldots z_n$.

But by the division algorithm, we can write $f(x) = (x-z_1)q_1(x) + r_1(x)$, where $\deg(r_1) < \deg(x-z_1) = 1$ or $r_1 = 0$. So $r_1$ is some constant. Thus $z_1$ is a root of a polynomial $f$ if and only if $(x-z_1)$ divides $f$. That means that $f$ factors completely:

$$f(x) = (x-z_1)(x-z_2)\ldots(x-z_n).$$

So $f$ factors into a product of $n$ non-units. Thus $f$ prime implies $f$ irreducible implies that $f$ has degree 1. Moreover, since a polynomial of degree 1 can only be factored if one of its factors is a constant, i.e. a unit, any polynomial of degree 1 is irreducible and thus prime. (Since we’re in a PID.)

Step 2. The modules $R/(x-z)^m$.

To use the PID module theorem, we have to figure out what the pieces described in it look like. So consider the ideal $I = ((x-z)^m)$. Then what does $R/I$ look like? Well, again leaning on the Euclidean algorithm, we know that any polynomial $f(x)$ can be written as

$$f(x) = q(x)(x-z)^m + r(x),$$

where $r = 0$ or $\deg(r(x)) < n$. That means that $f(x) + I = r(x) + I$. This means that for any coset $r + I$, we can choose to represent it by a polynomial with degree $< n$.

That is, $R/I = \{r + I \mid \deg(r) < n\}$. In fact, none of these cosets repeat. If $r + I = s + I$, then $r - s \in I$, but that can’t happen since $r - s$ would have degree $< n$. So $R/I$ essentially IS the set of all polynomials of degree $< n$.

More precisely, we can say that as a vector space, the set $\{1 + I, x + I, x^2 + I, \ldots x^{n-1} + I\}$ is a basis for $R/I$. This is true since every element of $R/I$ is uniquely of the form $a_0 + a_1x + \ldots a_{n-1}x^{n-1} + I$. But the basis I really want to use is $\{1 + I, (x-z) + I, (x-z)^2 + I, \ldots (x-z)^{n-1} + I\}$.

Now, it’s very easy to see how $(x-z)$ acts on this basis. $(x-z) \cdot (1 + I) = (x-z) + I$, $(x-z) \cdot ((x-z) + I) = (x-z)^2 + I$, etc. But at the end, $(x-z) \cdot ((x-z)^{n-1} + I) = (x-z)^n + I = 0 + I$. So $x-z$ sends each basis element to the next one, and kills the last one. That is, (for $n = 3$ for simplicity)

$$(x-z) \cdot v = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} v$$
\[ x \cdot v - z \cdot v = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} v \]
\[ x \cdot v = z \cdot v + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} v = \begin{bmatrix} z & 0 & 0 \\ 1 & z & 0 \\ 0 & 1 & z \end{bmatrix} v. \]

So \( R/(x - z)^n \) is described by that matrix.

Step 3. Understanding Cartesian Products.
This is the easy part.

**Theorem 2.** Let \( M \) and \( N \) be \( \mathbb{C}[x] \)-modules described by the matrices \( A \) and \( B \), respectively. Then \( M \times N \) is described by the matrix \[
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
\]

Proof. Define an isomorphism between the modules by writing elements \( m \) and \( n \) in \( M \) and \( N \) as vectors, and letting \( f((m, n)) = \begin{bmatrix} m \\ n \end{bmatrix} \). Then
\[
f(x \cdot (m, n)) = f(\lambda m, Bn) = \begin{bmatrix} Am \\ Bn \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = x \cdot f((m, n)).
\]

That the scalars work well is clear.

Step 4. Putting it all together.

**Theorem 3.** Let \( M \) be any \( n \times n \) matrix over \( \mathbb{C} \). Then \( M \) is similar to a unique matrix of the form \[
\begin{bmatrix}
J_{\lambda_1,a_1} & 0 & 0 & \cdots & 0 \\
0 & J_{\lambda_2,a_2} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & J_{\lambda_k,a_k}
\end{bmatrix},
\]
for some complex numbers \( \lambda_i \), where \( J_{\lambda,m} \) is the Jordan block described above. (E.g. \( J_{z,3} = \begin{bmatrix} z & 0 & 0 \\ 1 & z & 0 \\ 0 & 1 & z \end{bmatrix} \).) This matrix is unique up to permuting the \( \lambda_i \).

Proof. By the theorem for modules over a PID, the module described by \( M \) is isomorphic to a unique module of the form \( R/(x - \lambda_1)^{a_1} \times \cdots \times R/(x - \lambda_m)^{a_m} \). But that module is isomorphic to one described by the matrix \( A \). Thus, since the two are isomorphic, \( M \) must be similar to \( \text{THING} \).
\( \square \)

The matrix \( A \) above is called the Jordan canonical form of \( M \). It is the closest that \( M \) gets to being diagonalizable. Similarity is an equivalence relation on matrices, and the equivalence classes are then represented by the Jordan matrices.

Any \( \lambda \) that appears along the diagonal is an eigenvalue, and the number of times \( \lambda \) appears is called its algebraic multiplicity. The “type” of an \( R \)-module (as defined last time) is the same as counting all the algebraic multiplicities of the eigenvalues.

Ex. How many similarity classes of matrices are there whose only eigenvalue is 1, with algebraic multiplicity 5? Well, \( p(5) = 7 \), so there are 7 isomorphism classes of modules with type \((x - 1)^5\).
Thus there are 5 similarity classes of matrices:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}.
\]