1. \( x = 1 + r x + x^2 \)

From the quadratic formula, the roots of \( x^2 + r x + 1 = 0 \)
are \( r \pm \sqrt{r^2 - 4} \). There are no real roots for \( r \in (-2, 2) \); 

- There is one real root when \( r = \frac{-1}{2} \)
- and there are 2 real roots for \( |r| > 2 \).

For \( r = 2 \):

- The system has a saddle point.
- There is a saddle node bifurcation on either side of the \( x \) axis.
2. \( x = r x (1-x) \)

First, I solve for \( r_c \) (the critical value of \( r \)) by requiring:

\[ x = r x (1-x) \tag{1} \]

and \((x)' = [r x (1-x)]' \) or \( \dot{z} = r x (-1) + (1-x)r \) \tag{2}

Solving (1) for \( r \) gives \( r = \frac{1}{2} \) - \( x \). Plugging that in to (2), gives \( x^2 - 0 \Rightarrow r_c = 1 \).

Then, here are some characteristic trends:

\[
\begin{align*}
\text{r < 1} & \quad \text{(used r = 0.5)} \\
\text{r = 1} & \\
\text{r > 1} & \quad \text{(used r = 0.3)} \\
\end{align*}
\]

transcritical bifurcation
3. \[ x = x + \frac{rx}{1+x^2} \]

Again, we can solve for \( r \) first using:

(1) \(-x = \frac{rx}{1+x^2}\)

(2) \(x-x') = \frac{rx}{1+x^2} \cdot x' \cdot x''

From (1) we get \( r = -(1+x^2) \) or \( x^2 = -r - 1 \)

(2), explicitly, is: \[-1 = \frac{r(1+x^2)-(rx)(2x)}{(1+x^2)^2} \]

Solving this gives \( r_0 = -1 \).

\[ \text{bifurcation diagram:} \]

\[ \text{Subcritical pitchfork} \]
4. \[ \dot{N} = rN(1 - N/k) - \frac{HN}{(A + N)} \]

(a) When \( N = A \), the fish are harvested, so this is the maximum harvest date. So \( A \) is a sort of a relative measure of the size of the fish population.

(b) To nondimensionalize we need to rescale \( N \) and \( t \). We can think of \( N = \hat{N} \bar{N} \) (\( \hat{N} \) is dimensionless, \( \bar{N} \) is the dimensional scale) and \( t = \hat{t} \overline{t} \) (\( \hat{t} \) is dimensionless, \( \overline{t} \) is the dimensional scale). Our parameters are \( r, k, H \) and \( A \). We know \( k \) has units of population, so we introduce it for \( \hat{N} \). We will choose \( \overline{t} \) later to obtain the desired formulation.

LHS \[ \frac{d\hat{N}}{dt} = \frac{d(\hat{N}\bar{N})}{d(\hat{t}\overline{t})} = \frac{\bar{N} d\hat{N}}{\overline{t} dt} \]

RHS \[ r\hat{N}(1 - \frac{\hat{N}}{k}) - \frac{\hat{N}\bar{N}}{A + \hat{N}} \]
\[ = r\hat{N}k\left(1 - \frac{\hat{N}}{k}\right) - \frac{\hat{N}\bar{N}}{A + \hat{N}} \]
\[ = k \left[ r\hat{N}(1 - \hat{N}) - \frac{\hat{N}\bar{N}}{A + \hat{N}} \right] \]

so \[ \frac{k}{\overline{t}} \frac{d\hat{N}}{dt} = k \left[ r\hat{N}(1 - \hat{N}) - \frac{\hat{N}\bar{N}}{A + \hat{N}} \right] \]

\[ \frac{d\hat{N}}{dt} = r\hat{N}(1 - \hat{N}) - \frac{\hat{N}\bar{N}}{A + \hat{N}} \tag{\#} \]

we want \( \overline{t} = 1 \) so choose \( \overline{t} = 1/r \) (check units! Is this okay?)

we also want to rescale the denominator of the harvesting term. So multiply the whole fraction by \( \frac{\overline{t}k}{\overline{t}k} = \frac{1}{1/k} \)

Then \( \# \) becomes
\[ \frac{d\hat{N}}{dt} = \hat{N}(1 - \hat{N}) - \frac{\hat{N}\bar{N}}{A + \hat{N}} \]

for \( \bar{N} = \frac{1}{kr} \) and \( a = \frac{k}{A} \).
(c) We know when \( \frac{dN}{dt} = 0 \), \( N(1-N) = \frac{aN}{a+N} \)

We fixed points are the roots of the equation:

\[
N^2 + (a-1)N + (a-a) = 0
\]

\( N = 0 \) is always a fixed point. The other fixed points satisfy

\[
-(a-1) \pm \sqrt{(a-1)^2 - 4(a-a)}
\]

When the value under the radical is positive, the system has 2 fixed points. When the value under the radical is 0, the system has 1 real fixed point. For a in

\( S^1 \) the radical is negative, there is only 1 fixed point 0.

The critical value of \( a \) (at which the bifurcation occurs) is the value \( a_c \) s.t. \( (a-1)^2 - 4(a-a) = 0 \). Since \( a \) is positive, this occurs when \( a = 1 + 2\sqrt{2} \).

\( N=0 \) is an unstable fixed point.

When there are 2 fixed points, 0 is unstable and

\( (1-a)/2 \) is stable.

When there are 3 fixed points, 0 is unstable and the other 2 are stable.

(d) \( \frac{dN}{dt} = \frac{N}{a+N} - \frac{N}{a+N} = f(N) \)

Consider \( f'(N) = \frac{N(1-N) + (1-N)(a+N) - (a+N)-a-\frac{N}{a+N}}{(a+N)^2} \)

\( f'(0) = 1 - \frac{a}{a^2} \)

So when \( a = h \), \( f'(0) = 0 \) so that 0 is semi-stable.

If \( a > h \), or \( a < h \), the stability changes.

This is transcritical.

(e) I already showed that a occurs above. This is a saddle node bifurcation.
5. \( x = y - 2x; \ y = M + x^2 - y^2 \)

(a) The x-nullclines is given by \( y = 2x \). The y-nullclines is given by \( y = M + x^2 \). The appearance of the nullclines (and fixed points) depends on \( M \). There are fixed points when \( 2x = M + x^2 \) or \( x^2 - 2x + M = 0 \). This has the real root for \( M < 1 \) — this is \( M_0 \) (the bifurcation point).

Otherwise, \( x = \frac{2 \pm \sqrt{4 - 4M}}{2} \), which has no real roots when \( M > 1 \) (i.e., no fixed points) and 2 real roots for \( M < 1 \). So there are 3 nullcline cases:

- \( M > 1 \)
- \( M = 1 \)
- \( M < 1 \)

(b) The bifurcation occurs at \( M_0 = 1 \). It is a saddle node bifurcation.

(c) See attached phase portraits.
6. \[
x = y
\]
\[
y = \mu y + x - x^2 + xy
\]

(a) \[
J = \begin{bmatrix}
0 & y + 1
Y + x & M + x
\end{bmatrix}
\]
\[
J(0,0) = \begin{bmatrix}
0 & 1
1 & M
\end{bmatrix}
\]
\[
\text{evaluates } \quad M = \sqrt{M^2 + 4}
\]

But \(\sqrt{M^2 + 4} > M\) no matter what \(M\) is. So \(\frac{1}{2}(M + \sqrt{M^2 + 4}) > 0\) and \(\frac{1}{2}(M - \sqrt{M^2 + 4}) < 0\); i.e. the evaules are real with opposite signs and so \((0,0)\) is a saddle point for all \(M\).

(b) Any fixed point must have \(y = 0\) (x nullcline) so we look at the y nullcline when \(y = 0\): \(x - x^2 = 0\)

so the other fixed point is \((1,0)\).
\[
J(1,0) = \begin{bmatrix}
0 & 1
-1 & M+1
\end{bmatrix}
\]
\[
\text{evaluates } \quad -\frac{(M+1) + \sqrt{(M+1)^2 - 4}}{2}
\]

When \(M = 0\), there is the repeated evaules \(1 (= \frac{1}{2})\) and \((1,0)\) is an unstable node. When \(M > 1\) or \(M < -2\), there are 2 real evaules which can have opposite sign. When \(-2 < M < 1\) the Re part of the evaules are negative and so the fixed point is a stable spiral. For \(-1 < M < 1\) the Re parts are positive so \((1,0)\) is an unstable spiral. At \(M = 1\) the linearization predicts a center. This would need to be checked more.

(c) (e) See attached plots.

(d) From Maple, \(M = -8645\)
Exercise 5c. The following plots are generated from pp:ane6 using $\mu = -1, 0, 1, 2$. 
Exercise 6c. The following plots were generated from pplane6 using $\mu = -0.99, -0.75, -0.6, -0.5$. 
Exercise 6e. The following plots were generated from pplane6 using $\mu = -0.9, -0.8645, -0.8$. 