MATH 11 FALL 2008: LECTURE 15

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1. Complex Numbers

Recall the definition of complex numbers from Lecture 1.

Definition 1. *Complex numbers are elements of the set*

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R} \quad i = \sqrt{-1}\}.$$

Example 2. *The number*

$$z = 2 + 5i$$

is complex.

Definition 3. *Let* $z = a+bi \in \mathbb{C}$ *be a complex number. The* real part *of* z *is* a *and the* imaginary part *of* z *is* b, *which are denoted respectively by*

$$\mathrm{Re}(z) = a \qquad\qquad \mathrm{Im}(z) = b.$$

Remark 4. Note that $\mathrm{Im}(z) = b$, not bi. So both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are real numbers.

Example 5.

$$\mathrm{Re}(2 + 5i) = 2 \qquad\qquad \mathrm{Im}(2 + 5i) = 5$$

Remark 6. Using real and imaginary parts, complex numbers may be graphed on the complex plane, with a real axis and a complex axis, the latter keeping track of complex parts.

Definition 7. *The* magnitude *or* length *of* $z = a + bi \in \mathbb{C}$, *denoted* $|z|$, *is the distance from* z *to the origin* $0 + 0i$, *i.e.*

$$|z| = \sqrt{a^2 + b^2} \in \mathbb{R}.$$

Example 8.

$$|3 + 4i| = \sqrt{3^2 + 4^2} = 5$$
$$|-2 + i| = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$|i| = \sqrt{0^2 + 1^2} = 1$$
$$|-5| = \sqrt{(-5)^2 + 0^2} = 5$$

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Definition 9. *The* complex conjugate *of* $z = a + bi$ *is denoted* $\bar{z}$ *and is defined to be*

$$\bar{z} = a - bi.$$

Example 10.

$$\overline{3+4i} = 3 - 4i$$
$$\overline{3-4i} = 3 + 4i$$
$$\overline{-2+i} = -2 - i$$

2. Complex Arithmetic

2.1. Addition.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Example 11.

$$(3 + 4i) + (-2 + i) = 1 + 5i$$

2.2. Multiplication.

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Example 12.

$$(3 + 4i)(-2 + i) = -6 - 4 + 3i - 8i = -10 - 5i = -5(2 + i)$$

Remark 13. Note that

$$(a + bi)(a - bi) = a^2 + b^2 = |a + bi|^2$$

In general, we have

$$z\bar{z} = |z|^2 \in \mathbb{R}.$$

Theorem 14. *For any two complex numbers* z_1, z_2*,*

$$|z_1 z_2| = |z_1| \cdot |z_2|.$$

Proof. Go for it! Let $z_1 = a + bi$ and $z_2 = c + di$ and calculate away!

2.3. Division.

Example 15.

$$\frac{3+4i}{2+i} = \frac{3+4i}{2+i} \cdot \frac{2-i}{2-i} = \frac{10+5i}{4-i^2} = 2 + i$$

3. Geometry

3.1. **Addition.** Adding complex numbers corresponds to adding *vectors* in the plane. For example, $(3+4i)+(-2+i)=(1+5i)$ is like

$$(3,4)+(-2,1)=(1,5),$$

which we can draw in $\mathbb{R}^2$. A useful theorem concerning the geometry of complex addition is the following.

Theorem 16 (Triangle Inequality). *For any complex numbers z and w,*

$$|z+w| \leqslant |z|+|w|.$$

3.2. **Multiplication.**

3.2.1. *Polar Coordinates.* In order to understand the geometry of complex multiplication, it's useful to use another coordinate system for complex numbers, i.e. another way of describing what complex numbers are.

We can think of the coordinates $z = a+bi$ of a complex number as a set of instructions: originating at the origin (!), move a spaces on the real axis, right or left depending on the sign of a, and then move b spaces parallel to the imaginary axis, again up or down depending on the sign of b. In this way every point in the complex plane has unique coordinates, and we have a complete set of instructions detailing how to move from the origin to any point in the complex plane.

However this procedure isn't unique! Polar coordinates offer alternative instructions for movement from the origin to any point in the complex plane. The idea is to first rotate an angle θ to aim directly at the target point $z \in \mathbb{C}$. Then, move directly to the point z, which is by definition a distance $|z|$ from the origin. Instead of moving left and right, in *rectangular* coordinates, our movement in polar coordinates is direct!

To make this idea more precise, we claim that every point z in the complex plane can be specified by the data R, θ, where R is the distance from z to the origin, and θ is the angle subtended by z from the real axis, measured in radians counterclockwise.

Indeed, suppose $z = a+bi$. Then as noted above, z is the distance $|z| = \sqrt{a^2+b^2}$ from the origin. Thus

$$R = |z| = \sqrt{a^2+b^2}.$$

If z is an angle θ subtended from the real axis, then

$$a = R\cos\theta \qquad\qquad b = R\sin\theta.$$

Therefore

$$\theta = \tan^{-1}(b/a).$$

This allows us to write

$$z = a+bi = R(\cos\theta + i\sin\theta).$$

Thus we have an explicit algorithm to produce polar coordinates for any $z \in \mathbb{C}$ given in rectangular coordinates, and vice versa.

3.2.2. *Multiplication via polar coordinates.* Let's now discover the geometry of complex multiplication via polar coordinates. Let $z_1 = R_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = R_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= R_1(\cos\theta_1 + i\sin\theta_1) \cdot R_2(\cos\theta_2 + i\sin\theta_2) \\ &= R_1 R_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= R_1 R_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

Awesome! We now have a direct geometric interpretation of multiplication of complex numbers: their *lengths* multiply and their *angles* add!