MATH 11 FALL 2008: LECTURE 18

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1. TAYLOR SERIES

In our last lecture we visited the idea of approximating a function with a polynomial, a Taylor polynomial, and via Taylor's theorem we have precise knowledge of the error in this approximation.

Example 1. *Let's calculate the* n^{th} *Taylor approximation of* $f(x) = e^x$ *at the basepoint* 0*. Well* $\frac{d}{dx}e^x = e^x$*, so*

$$f^{(n)}(x) = e^x,$$

for any n*. Thus*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(n)}(0)}{k!}(x-0)^k = \sum_{k=0}^{n} \frac{e^0}{k!}x^k = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$

The error is defined by

$$\begin{aligned} E_n(x) &= f(x) - P_n(x) \\ &= e^x - \left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}\right). \end{aligned}$$

By Taylor's theorem, there exists some number c *between* 0 *and* x *such that*

$$E_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-0)^{n+1} = \frac{e^c}{(n+1)!}x^{n+1}.$$

Therefore

$$e^x = P_n(x) + E_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}.$$

For instance, if $x = 1$ *and* $n = 9$*, then*

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{9!} + \frac{e^c}{10!},$$

for some $0 < c < 1$*. The error term* $E_9(1)$ *is a function of only* c*,* $E_9(1) = e^c/10!$*. Since* e^x *is an increasing function,*

$$E_9(1) < \frac{e^1}{10!} < 10^{-6}.$$

Moreover, for any fixed x*, it can be shown that*

$$\lim_{n\to\infty} E_n(x) = \lim_{n\to\infty} \frac{e^c x^{n+1}}{(n+1)!} = 0.$$

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Therefore

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

This is called the Taylor series *of* e^x *at basepoint* 0.

Definition 2. *Let* f *have derivatives of all orders at the point* a *in the domain of* f. *The* Taylor series *of* $f(x)$ *at basepoint* a *is given by*

$$P(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Example 3. *We have the following Taylor series.*

(1) $e^x = 1 + x + x^2/2 + \cdots$ *for all* x.
(2) $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \cdots$ *for all* x.
(3) $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots$ *for all* x.
(4) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ *for* $|x| < 1$.
(5) $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$ *for* $-1 < x \leqslant 1$.

2. Infinite series

To understand any of the above Taylor series, we have to come to terms with the idea of an infinite sum.

Definition 4. *An* infinite series *is a sum of the form*

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + a_3 + \cdots,$$

where a_k *is a number (real or complex) for each* $k = 0, 1, \ldots$.

Definition 5. *The series* $\sum_{k=0}^{\infty} a_k$ *is said to* converge *to the number* L *if*

$$\lim_{n\to\infty} \left(\sum_{k=0}^{n} a_k \right) = L.$$

Remark 6. The finite sum

$$\sum_{k=0}^{n} a_k$$

is called the n^{th} *partial sum* of the infinite series

$$\sum_{k=0}^{\infty} a_k.$$

Example 7. *Consider the series*

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k.$$

Does this series converge? We need to consider the partial sums of our series.

We claim

$$\sum_{k=0}^{n} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{2^{n+1} - 1}{2^n} = 2 - \frac{1}{2^n}.$$

Indeed, we can see this using induction. For the base case, note that

$$\sum_{k=0}^{0} \frac{1}{2^k} = \frac{1}{2^0} = 1 = \frac{2^1 - 1}{2^0}.$$

For induction, assume

$$\sum_{k=0}^{n} \frac{1}{2^k} = \frac{2^{n+1} - 1}{2^n}.$$

We compute

$$\begin{aligned}
\sum_{k=0}^{n+1} \frac{1}{2^k} &= \sum_{k=0}^{n} \frac{1}{2^k} + \frac{1}{2^{n+1}} \\
&= \frac{2^{n+1} - 1}{2^n} + \frac{1}{2^{n+1}} \\
&= \left(\frac{2^{n+1} - 1}{2^n}\right) \cdot \left(\frac{2}{2}\right) + \frac{1}{2^{n+1}} \\
&= \frac{2^{n+2-2}}{2^{n+1}} + \frac{1}{2^{n+1}} \\
&= \frac{2^{n+2} - 1}{2^{n+1}}.
\end{aligned}$$

Therefore our proof by induction holds, and the claim is true.

We can now use this to determine the convergence of our series.

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \lim_{n \to \infty} \left(\sum_{k=0}^{n} \frac{1}{2^k} \right) = \lim_{n \to \infty} \left(2 - \frac{1}{2^n} \right) = 2 - 0 = 2.$$

Therefore our series converges, and it converges to 2.