MATH 11 FALL 2008: LECTURE 2

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1. Announcements

(1) Homework 1 update re. use induction.
(2) Collaborative homework: working in groups is groovy, write up independently.
(3) Both DK and AB office hours are open to all sections.

2. Numbers continued

Definition 1. *Here we set notation for the integers, rational numbers, real and complex numbers, respectively.*

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$
$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$$
$$\mathbb{R} = \{\textit{All decimals}\}$$
$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}, i = \sqrt{-1}\}$$

Remark 2. There are a plethora of interesting and important constructions of the real numbers, including limits of rationals, Dedekind cuts and decimal expansions, to name the most common. These may be explored further in an analysis course. For our immediate purposes, it suffices to note that a number is a real number if and only if it may be expressed (not uniquely, e.g. $.999\cdots = 1$) as a (not necessarily terminating, e.g. $e = 2.718281828\cdots$) decimal.

3. Induction continued

Example 3. *Prove*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

for all $n \in \mathbb{N}$.

Proof. First, we consider the base case. Note that

$$1 = \frac{2}{2}$$

and thus equation 1 holds when $n = 1$.

Now, assume that equation 1 holds for a number $k \in \mathbb{N}$, i.e.

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

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Under this hypothesis, we want to show 1 holds for $k+1$. Thus, we want to show that

$$1+2+3+\cdots+(k+1)=\frac{(k+1)((k+1)+1)}{2}.$$

So, we compute.

$$\begin{aligned}1+2+3+\cdots+k+1 &= (1+2+3+\cdots+k)+k+1\\ &= \frac{k(k+1)}{2}+k+1\\ &= \frac{k(k+1)+2(k+1)}{2}\\ &= \frac{k^2+3k+2}{2}\\ &= \frac{(k+1)(k+2)}{2}\end{aligned}$$

□

Example 4. *After removing an arbitrary square from a $2^n \times 2^n$ checkerboard, the remaining board can be covered by non-overlapping trominoe tiles.*

4. Method of proof: contradiction

Theorem 5. *$\sqrt{2}$ is irrational.*

Proof. Suppose that $\sqrt{2} \in \mathbb{Q}$. Then let $\sqrt{2} = a/b$ be its *reduced* fraction expression, i.e. a and b have no common factors. Then we compute

$$2 = a^2/b^2 \Rightarrow 2b^2 = a^2.$$

Therefore a^2 is even, and hence a is itself even, i.e. $a = 2k$ for some integer k.

Using this, we have

$$\sqrt{2} = \frac{2k}{b} \Rightarrow 2b^2 = 4k^2 \Rightarrow b^2 = 2k^2.$$

Hence b^2 and b are even. But then a and b have a common factor (2!), and hence a/b is not reduced, contradicting our assumption. Therefore our assumption is false. Therefore $\sqrt{2}$ is not rational. □

Example 6. *There exist irrational numbers a and b such that a^b is rational.*

Proof. Is $\sqrt{2}^{\sqrt{2}}$ rational? If yes, then we are done. Otherwise, let $a = \sqrt{2}^{\sqrt{2}}$ and let $b = \sqrt{2}$. Then

$$a^b = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^2 = 2$$

Therefore, in either case, we have a pair of irrational numbers a, b such that a^b is rational. □

Note that this example illustrates the non-constructive nature, typical of proofs by contradiction. We have not generated a particular example, but we may conclude that one necessarily exists.