MATH 11 FALL 2008: LECTURE 3

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1. Announcements

(1) Handouts: ϵ, δ proofs.
(2) Homework 2 due Wed. Sept 10, not Tuesday as usual.

2. Limits

The concept of *limit* is of fundamental importance in the study of calculus. Indeed, if one attempts to divide the world of mathematical knowledge into pre- and post-calculus (which is not necessarily a valid distinction), the central philosophical distinction is the notion of limit.

We begin our discussion with limits of real valued functions. What is the limit of a (real valued) function? We want to give meaning to the expression "the limit as x approaches x_0 of the function $f(x)$ is equal to L," which is denoted

$$\lim_{x \to x_0} f(x) = L. \tag{1}$$

Perhaps we would like this to indicate that as x "gets closer" to x_0, $f(x)$ gets closer to L. But this statement is insufficiently precise. How close is close enough? As close as we want. So perhaps we desire (1) to imply that $f(x)$ is *arbitrarily* close to L, for x sufficiently close to x_0.

This is now a statement that we may attempt to translate into mathematics. First, what is "close?" This is of course a relative adjective concerning distance. What is distance?

Definition 1. *The* distance *between two real numbers a and b is given by*

$$|a - b| = \textit{Distance from } a \textit{ to } b$$

Recall the definition of absolute value.

Definition 2. *The* absolute value *of a real number x is given by*

$$|x| = \begin{cases} x & \textit{if } x \geq 0 \\ -x & \textit{if } x < 0 \end{cases}$$

Since we're discussing absolute values, we record the following fact for posterity.

Proposition 3. *For any real numbers a and b, the following* triangle inequality *holds.*

$$|a + b| \leq |a| + |b|$$

We now have the tools to make precise the definition of limit.

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Definition 4. *Let $f(x)$ be a real valued function, let $x_0 \in \mathbb{R}$ be a point, not necessarily in the domain of f, and let L be a real number. The* limit of $f(x)$ as x approaches x_0 is equal to L *if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$0 < |x - x_0| < \delta \textit{ implies } |f(x) - L| < \epsilon.$$

In this case we write

$$\lim_{x \to x_0} f(x) = L.$$

Remark 5. First, note the correspondence between this definition and our intuitive first approximation. Second, note the geometric nature of this definition. Lastly, note the importance of distinction between x and x_0; the distance between the two is required to be strictly positive.

Example 6. *Prove* $\lim_{x \to 1} 8x - 3 = 5$.

Before we begin our proof, lets first study the problem. We must show for each $\epsilon > 0$ there is $\delta > 0$ such that $0 < |x - 1| < \delta$ implies $|(8x - 3) - 5| < \epsilon$.

So, we perform exploratory computations.

$$\begin{aligned} |(8x-3)-5| = |8x-8| &< \epsilon \\ \iff 8|x-1| &< \epsilon \\ \iff |x-1| &< \epsilon/8 \end{aligned}$$

Our quest to discover δ seems to be complete. We have found a number, namely $\delta = \epsilon/8$, such that $|x - 1| < \delta$ implies $|(8x - 3) - 5| < \epsilon$. We now formalize this proof.

Proof. Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/8$. Then if

$$0 < |x - 1| < \delta = \epsilon/8$$

Then

$$|(8x - 3) - 5| = |8x - 8| = 8|x - 1| < 8\delta = 8 \cdot \epsilon/8 = \epsilon.$$

Therefore $|f(x) - L| < \epsilon$, as desired. $\square$

Remark 7. Note the double edged practice of covering one's tracks displayed in this proof. As written, the choice of δ in the above proof may appear initially unmotivated. On the other hand, the proof is rather concise without inclusion of exploratory calculations.